Introduction to the mathematics of the general relativity

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Contents

List of Figures

Preface

Chapter 1 is dedicated to a very brief introduction to relativity. 1.1 is inspired by [21] and [12], 1.2 by [**?**] and [29] and 1.3 by [31].

Chapter 1

From special relativity to Einstein equations

1.1 Special relativity

Newton in his theory of gravitation(1684) considered an absolute frame and then he stated that the laws of physics (including the gravitation) are the same in all frames which are in uniform translation with respect to the absolute one. But, at the end of XIXth century, the result of the experience of Michelson-Morley, interpreted by Lorentz, showed that the light propagate isotropically and then contradicted the composition of speed proposed by Newton. In 1905, Einstein proposed his theory in order overcome this issue. In fact the fact that the speed of light is same in any directions and finite impose a relative time to each observer as shown by the following famous thought experiment: A flash light occurs at time $t = 0$ in the middle (M) of a car which moves at speed v, see figure 1.1.

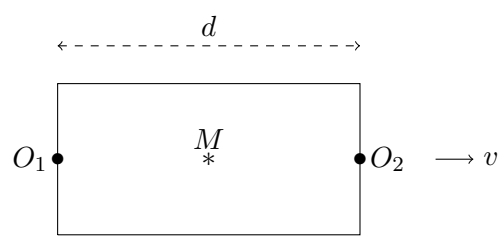

Figure 1.1: Relativity of time

Denoting by c the speed of light, the observer O_i receives the flash at time $t_i = \frac{d/2}{c \pm v}$. Hence in there inner frame, for each observer the flash light has been emitted at different times. Hence Einstein proposed the following postulate.

Postulates of the special relativity:

1. There is no absolute time, time is relative to the observer. There is no absolute frame.

2. Light propagates isotropically in any frame at the absolute speed c.

3. No particle or observer travels faster than light.

Hence to each spacial coordinates system is attached a fourth coordinates of time which is proper to the frame. Now space and time are linked and relative to the frame.

Since there is no more absolute frame, it is important to understand how we pass from a frame to an other.

Let $S(t, x, y, z)$ and $S'(t', x', y', z')$ two frames. Since a particle, which is subject to no exterior force, is in uniform motion, i.e. travel along straight line, hence the straight line of each referential should be in bijection. Thanks to the fundamental theorem of affine geometry, see §2.6 of [3], we pass from S to S' through a linear transformation. In fact it should be only an affine transformation, but we can assume that the two origins correspond, later we will say that each observer agree on the fact that the origin of their frame is the same event.

A photon must satisfies in S and S' the following equations

$$-c^2dt^2 + dx^2 + dy^2 + dz^2 = 0$$

and

$$-c^2dt'^2 + dx'^2 + dy'^2 + dz'^2 = 0.$$

Here, we briefly introduce the Minkowsky space $\mathbb{R}^{3,1}$ which corresponds to $\mathbb{R}^4(t, x, y, z)$ with the following metric $\|.\|^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$. The set of null vector is called light-cone, since it is where travel photon emitted from the origin. A vector $\vec{v}$ is said space-like (resp. time-like or light-like) if $\|\vec{v}\| > 0$ (resp. < 0 or $= 0$). The time-like vector correspond to speed of particle passing through the origin. The set of time-like vector has two connected component. We choose one as future, usually the one with $t > 0$ is future, and the other past. The Minkowsky space is a representation of space-time, it is usually attached to an observer. A point is called *an event*. The history of a moving particle in space time is a curve called *the world line* of the particle. Two events E_1 and E_2 are said *simultaneous* in a given representation if $t(E_1) = t(E_2)$. As we will see later this notion depends of the observer that is to say of the chosen frame.

Exercise 1.1 Show that if $A \in GL(4)$ preserves the light-cone, i.e. the set of points defined by $x^2 + y^2 + z^2 = ct^2$, then there exists $\lambda \in \mathbb{R}$ such that $\|Av\|^2 = \lambda\|v\|^2$ for all $v \in \mathbb{R}^4$ where $\|v\|^2 = -c^2v_0^2 + \sum_{i=1}^3 v_i^2$.

Thanks to the previous exercise and using the fact the transformation between the two frame must send particles on particles, i.e. future time-like vector onto future time-like vector, there exists $k > 0$ such that

$$dx^2 + dy^2 + dz^2 - c^2dt^2 = k(dx'^2 + dy'^2 + dz'^2 - c^2dt'^2).$$

Hence, up to dilate the coordinates of one frame, the transformation between the two frames is an isometry of Minkowsky space, usually called Lorentz transformation.

To simplify, let assume that S' is uniformly translated along the x-axis of S, as in figure 1.2.

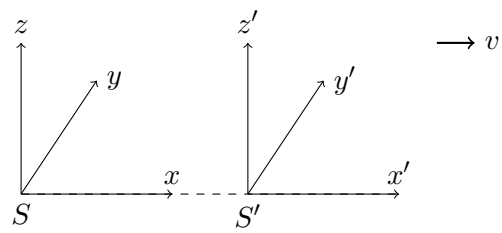

Figure 1.2: Two frames in translation

We temporally omit the coordinates y, z, y' and z'. We pass from the (t, x) coordinates to the (t', x') through an element $A \in O(1, 1)$, i.e. a $2 * 2$ matrix A which satisfies

$${}^tAGA = G,$$

where

$$G = \begin{pmatrix} -c^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

One easily check that there exists $u \in \mathbb{R}$ such that

$$A = \begin{pmatrix} \cosh(u) & \frac{1}{c}\sinh(u) \\ c\sinh(u) & \cosh(u) \end{pmatrix}.$$

As the origin of the frame S' moves at speed v, its coordinates satisfies

$$\frac{x}{t} = v,$$

but since

$$\begin{pmatrix} t \\ x \end{pmatrix} = A^{-1}\begin{pmatrix} t' \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh(u) & -\frac{1}{c}\sinh(u) \\ -c\sinh(u) & \cosh(u) \end{pmatrix}\begin{pmatrix} t' \\ 0 \end{pmatrix},$$

we get that

$$v = -c\tanh(u),$$

which gives

$$\begin{cases} \cosh(u) = \left(1 - \left(\frac{v}{c}\right)^2\right)^{-\frac{1}{2}} \equiv \gamma, \\ \sinh(u) = -\frac{v}{c}\gamma. \end{cases}$$

Finally in our special case the matrix that permits to pass from S' to S is

$$L_v = \gamma \begin{pmatrix} 1 & -\frac{1}{c^2}v \\ -v & 1 \end{pmatrix}.$$

Speeds composition law

We consider a frame S_2 which moves uniformly with speed v_2 along the x-axis of a frame S_1 which itself moves uniformly with speed v_1 along the x-axis of a frame S. Then the matrix which permits to pass from S_2 to S is

$$L_{v_1}L_{v_2} = \gamma_1\gamma_2 \begin{pmatrix} 1 + \frac{v_1 v_2}{c^2} & \frac{-v_2 - v_1}{c^2} \\ -v_2 - v_1 & 1 + \frac{v_1 v_2}{c^2} \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\frac{v}{c^2} \\ -v & 1 \end{pmatrix}.$$

Hence by identification, the speed of the origin of S_2 into S is given by

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}.$$

Exercise 1.2 Show that if $|v_1| \leq c$ and $|v_2| \leq c$ then $|v| \leq c$.

Simultaneity and contraction of length

Consider two observers at the same point. But one is moving with speed v we respect to the other. Then we deduce the frame of the second observer with respect to the frame of the first through a transformation as L_v. Hence superposing the two frame we obtain figure 1.3.

We sse that under the action of L_v the axes of the blue frame become closer to the ligth-cone in the future direction, this transformation is also know as *a Lorentzian boost*.

Then we consider a rod, of length l, with respect to the first frame. When the second observer want to measure the size of the rod he has to note simultaneously the position of the extremity. Drawing the world line of the to extremity in figure 1.4.

We observed that the rod is smaller in the blue frame, more precisely the second observer measure $l' = \frac{l}{\gamma} < l$. Indeed if the first observer measure the rod between $(0, \delta)$ and $(0, \delta + l)$, second will measure between (t', δ') and $(t', \delta' + l')$. Those two last events correspond in the first frame to two events $(t'', \delta) = L_v(t', \delta')$ and $(t''', \delta + l) = L_v(t', \delta' + l')$,

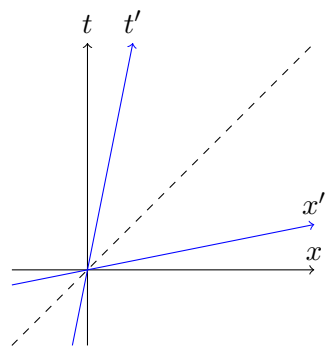

Figure 1.3: Two Observers with some relative speed

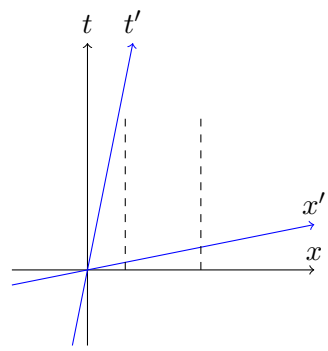

Figure 1.4: A rod view by two observers

indeed the extremity of the rod don't move in the first frame. But The time is the not the same, since two simultaneous event in the second frame have no reason to be simultaneous (and aren't except if $v = 0$) in the first frame. By linearity

$$(t''' - t'', l) = L_v(0, l')$$

which proves that $l = \gamma l'$.

Exercise 1.3 In fact, on figure 1.4, l' "seems" bigger...Draw unit length of each axis for $v = \frac{c}{2}$ to convince yourself it is indeed smaller. What happens if the rod is fixed with respect to the second frame?

The proper time of a particle

Let consider a moving particle, its world line is a curve parametrized by $\tau \mapsto \alpha(\tau)$. We assume that $\left(\frac{d\alpha}{d\tau}(\tau)\right)_t > 0$ and $\left\|\frac{d\alpha}{d\tau}(\tau)\right\| = -c^2$. The first assumption refers to some choice of orientation and the second to some normalization of the parametrization (such as arclength). Now τ is fixed up to a constant, it is called *the proper time* of the particle.

Let set

$$\alpha(\tau) = \begin{pmatrix} t(\tau) \\ x(\tau) \end{pmatrix},$$

and

$$\vec{u} = \frac{d\alpha}{d\tau} = \gamma \begin{pmatrix} 1 \\ v \end{pmatrix},$$

where $\gamma = \frac{dt}{d\tau}$ and $v = \frac{dx}{dt}$. But $\|\vec{u}\|^2 = -c^2$ gives

$$-c^2\gamma^2 + \gamma^2 v^2 = -c^2$$

hence

$$\gamma = \frac{c}{\sqrt{c^2 - v^2}}.$$

We remark that $\frac{dt}{d\tau} = \gamma \geq 1$, which means that the proper time of the particle decrease when it moves. Which leads to the famous **twin paradox**. The interpretation of this fact should be done carefully, the proper time of which we speak is the one measure in the frame of the brother which stays "fixed". But from the point of view of the "traveling" twin, that is the "fixed" twin who is moving. Hence which is the younger of the two? See [23] for more details.

Exercise 1.4 Solve the twin paradox by drawing the world line of the two bothers and their simultaneous events.

To finish this section, let us give an interpretation (due to Robb 1936) of the proper time. Assume we have two observers P and Q at the same place but different time and P send a light signal to Q which is reflect by a mirror in E as in the following figure.

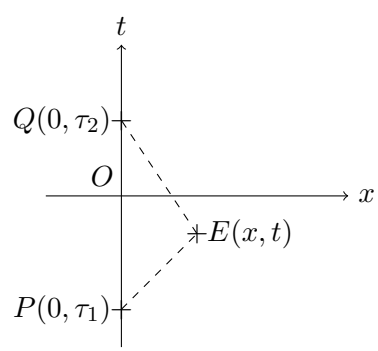

Figure 1.5: Interpretation of proper time

The time τ_1 and τ_2 depends on the observer but the quantity $\|OE\|^2$ is invariant by any change of frame.

$$\|OE\|^2 = -c^2t^2 + x^2$$

But $x = c(t-\tau_1) = c(\tau_2 - t)$ which leads to $x = c\frac{\tau_2-\tau_1}{2}$ and $t = \frac{\tau_1+\tau_2}{2}$. Hence

$$\begin{aligned} \|OE\|^2 &= -c^2\left(\frac{\tau_1+\tau_2}{2}\right)^2 + c^2\left(\frac{\tau_2-\tau_1}{2}\right)^2 \\ &= -c^2\tau_1\tau_2 \end{aligned} \tag{1.1}$$

Even each τ_i depends on the observer, but their product is an absolute quantity. Moreover if O and E are simultaneous, i.e. t=0, then the spatial distance is equal to the half-time that the light spends to go to one observer to the other, $|OE| = \|OE\| = -c\tau_1 = \frac{\tau_2-\tau_1}{2}c$.

Equivalence of mass and energy

To any particle we can associate a mass at rest (or proper mass) denoted m_0, then we get a momentum $\vec{p} = m_0\vec{u} = m_0\gamma\begin{pmatrix} 1 \\ v \end{pmatrix} = \begin{pmatrix} m \\ mv \end{pmatrix}$, with $m = m_0\gamma = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$. Hence the mass seen by an observer with speed v with respect to the particle is different of m_0, moreover $m \to +\infty$ as $v \to c$.

If the particle is submitted to a force $\vec{f}$, we have

$$\vec{f} = \frac{d\vec{p}}{d\tau} = m_0\frac{d\vec{u}}{d\tau} = \begin{pmatrix} \frac{dm}{d\tau} \\ \frac{d(mv)}{d\tau} \end{pmatrix} = \begin{pmatrix} \gamma\frac{dm}{dt} \\ \gamma\frac{d(mv)}{dt} \end{pmatrix} = \begin{pmatrix} f_0 \\ \gamma f_c \end{pmatrix}, \tag{1.2}$$

where f_c is the classical force. Moreover, as $\|\vec{u}\|^2 = -c^2$ we have $\langle\frac{d\vec{u}}{d\tau}, \vec{u}\rangle = 0$, which gives $\langle\vec{f}, \vec{u}\rangle = 0$, that is to say

$$\left\langle \begin{pmatrix} f_0 \\ \gamma f_c \end{pmatrix}, \begin{pmatrix} \gamma \\ \gamma v \end{pmatrix} \right\rangle = 0$$

hence $f_0 = \frac{\gamma}{c^2}f_c v$, which finally gives

$$\vec{f} = \begin{pmatrix} \frac{\gamma}{c^2}f_c v \\ \gamma f_c \end{pmatrix}.$$

Hence plugging this into (1.2), we get

$$\gamma\frac{dm}{dt} = f_0 = \frac{\gamma}{c^2}f_c v$$

which gives

$$c^2 dm = f_c dx.$$

Hence we remark that we need an infinite classical force to accelerate a particle to the speed of light,

$$W = \int f_c\, dx = c^2(m - m_0).$$

It is natural to think that the energy of a particle at rest is proportional to its mass, i.e.

$$E = km_0.$$

Now for an observer in motion with respect to the particle, the energy is equal to $E + E_c$, where E_c is the kinetic energy, but should be still proportional the mass (observed), hence

$$km = km_0 + E_c.$$

But we know that the difference of kinetic energy must be equal to the work of forces, hence

$$k(m - m_0) = W = c^2(m - m_0),$$

which leads to $k = c^2$ and

$$E = mc^2.$$

1.2 Electromagnetism in Minkowsky space

Now we comeback to a situation where the space has dimension 3 and we normalize the speed of light to $c = 1$, which leads us to study more care fully the space $\left(\mathbb{R}^4, \eta\right)$ where $\eta = -dx_0^2 + \sum_{i=1}^{3} dx_i^2$, denoted by $\mathbb{R}^{3,1}$.

Definition 1.2.1 *Let $v \in \mathbb{R}^{3,1}$, we say that*

- *v is of timelike if $\eta(v, v) < 0$,*
- *v is of lightlike if $\eta(v, v) = 0$,*
- *v is of spacelike if $\eta(v, v) > 0$.*

We denote by C the space of lightlike vector. Finally a sub manifold M of $R^{3,1}$ is said of

- *of space-type if TM consists only of spacelike vectors,*
- *of timelike if TM consists only of timelike vectors.*

The isometries of $\mathbb{R}^{3,1}$ are maps $F : \mathbb{R}^4 \to \mathbb{R}^4$ such that $F^*(\eta) = \eta$. There are translations, i.e. $F(x) = x + a$ where $a \in \mathbb{R}^4$, classical isometies of space $F(x) = Ax$ where $A \in GL_4(\mathbb{R})$ is of the form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}$$

where $A' \in SO(3)$. And finally some Lorentzian boost $F(x) = Ax$ where $A \in GL^4(\mathbb{R})$ is of the form

$$A = \begin{pmatrix} \cosh(u) & \sinh(u) & 0 & 0 \\ \sinh(u) & \cosh(u) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now we consider the electromagnetic field as a 2-form[1] on $\mathbb{R}^{3,1}$,

$$F = \frac{1}{2} F_{ab}\, dx^a \wedge dx^b,$$

with $F_{ab} = -F_{ba}$.
Let $I : \{x \,|\, x_0 = t\} \to \mathbb{R}^4$ the canonical inclusion map. Then we set $E = I^* \left(\iota_{\frac{\partial}{\partial x^0}} F \right)$ the electric field and $B = \star_3 \left(I^*(F) \right)$ the magnetic filed, where $\star_3$ is the standard Hodge star into $\mathbb{R}^3$. Denoting[2] $E = E_i\, dx^i$ and $B = B_i\, dx^i$, we can summarized the definition as follows

$$F_{|x_0=t} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}.$$

We denote by δ the adjoint of the exterior differential, which is defined by

$$\int \langle d\alpha, \beta \rangle dv = \int \langle \alpha, \delta\beta \rangle\, dv$$

for all $\alpha \in \Omega^1_c$ and $\beta \in \Omega^2_c$, where $dv = dx^0 \wedge \cdots \wedge dx^4$ and the scalar product of two form a and b of the same degree is defined by

$$\langle a, b \rangle = \eta^{i_1 j_1} \ldots \eta^{i_k j_k} a_{i_1 \ldots j_k} b_{j_1 \ldots j_k}$$

and the Hodge star by

$$\langle a, b \rangle dv = a \wedge \star b.$$

Exercise 1.5 Show that $\delta = \star d \star$ and deduce that

$$(\delta F)_a = -\eta^{bc} F_{ab,c}.$$

Maxwell's equations :

[1] The reader who is not very familiar with differential forms, will find a nice introduction in [8] and something more complete in [19].

[2] Here, as in the rest of the notes, we use the Einstein summation convention, when indices are repeated they are sum.

In a medium with density charge ρ and current J, E and B satisfies the classical Maxwell's equations

$$\begin{cases} \mathrm{div}(B) = 0 \\ \frac{\partial B}{\partial t} + \mathrm{curl}(E) = 0 \\ \mathrm{div}(E) = \rho \\ -\frac{\partial E}{\partial t} + \mathrm{curl}(B) = J \end{cases}$$

where E and B are seen as vector field depending on time.

Exercise 1.6 Show that Maxwell's equations in our formalism reduce to

$$\begin{cases} dF = 0 \\ \delta F = j \end{cases},$$

where $j = \rho dx^0 + J_i dx^i$.

Constraint equations :

In the following we will be mainly interested in the vacuum case where $j = 0$. We look for initial conditions on E^0 and B^0 in $\Omega^1(\mathbb{R}^3)$ to solve the following equations

$$\begin{cases} dF = 0 \\ \delta F = 0 \\ F_{|x_0=0} = F_0 \end{cases}, \tag{ME}$$

where

$$F_0 = \begin{pmatrix} 0 & E_1^0 & E_2^0 & E_3^0 \\ -E_1^0 & 0 & B_3^0 & -B_2^0 \\ -E_2^0 & -B_3^0 & 0 & B_1^0 \\ -E_3^0 & B_2^0 & -B_1^0 & 0 \end{pmatrix}.$$

As a necessary condition we observe that

$$0 = I^*(dF) = dI^*(F) = d(\star_3 B)$$

and

$$0 = (\delta F)_0 = (\mathrm{div} E),$$

where we used exercise 1.5 in the last equality. Restricting those equations to $x_0 = 0$ we get

$$\begin{cases} \mathrm{div}(B^0) = 0 \\ \mathrm{div}(E^0) = 0, \end{cases} \tag{CE}$$

this last system is known as the constraint equations. The following theorem assert that the condition is sufficient.

Theorem 1.2.2 *Let $E^0, B^0 \in \Omega^1(\mathbb{R}^3)$, respectively C^4 and C^3, which satisfies (CE) and such that*

$$|B_i^0| = O(r^{-2-\varepsilon}) \text{ for all } i \text{ and some } \varepsilon > 0, \tag{1.3}$$

then (ME) admits a unique solution on $\mathbb{R}^4$.

Proof of theorem 1.2.2:

As $\mathbb{R}^4$ is simply connected, $dF = 0$ implies that there exists $A \in \Omega^1(\mathbb{R}^{3,1})$ such that $F = dA$. We are gooing to look for A in a first time. Since $\delta F = 0$, A must satisfies $\delta dA = 0$. But A defined up to some differential df since $d(A + df) = dA$. This is known as a Gauge choice. Hence we assume that we can choose A such that $\delta A = 0$. So we in fact look for A a solution of

$$\Box A = (d\delta + \delta d)A = 0,$$

where $\Box$ is Hodge Laplacian on form, we denoted $\Box$ rather than Δ because the metric is Lorentzian, which is also justifies by the following exercise.

Exercise 1.7 Show that each component of A satisfies the classical wave equation.

Thanks to theorem 3 of §2.4 of [10], there is a solution of

$$\begin{cases} \Box A = 0 \\ A_{|x_0=0} = A^0 \\ (A_{,0})_{|x_0=0} = A^1 \end{cases}, \tag{1.4}$$

for any A^0, A^1 which are seen as restriction of element of $\Omega^1(\mathbb{R}^4)$ to $\{x_0 = 0\}$ and which are at least C^3.

Hence we now concentrate on how to choose A^0 and A^1 to get a solution of our problem.

Claim 1: There exists A^0 and A^1, such that a solution of (1.4) must satisfies

$$\begin{cases} \Box A = 0 \textbf{ on } \mathbb{R}^4 \\ \delta A = 0 \textbf{ on } \{x \,|\, x_0 = 0\} \\ (\delta A)_{,0} = 0 \textbf{ on } \{x \,|\, x_0 = 0\} \\ dA = F_0 \textbf{ on } \{x \,|\, x_0 = 0\} \end{cases}. \tag{1.5}$$

Proof of the Claim 1:

Since B^0 is given and $d(\star_3 B^0) = 0$, since $\mathbb{R}^3$ is simply connected, there exists $\tilde{A} \in \Omega^1(\mathbb{R}^3)$ such that

$$d\tilde{A} = \star_3 B^0.$$

Let $f : \mathbb{R}^3 \to \mathbb{R}$ a solution of

$$\Delta f = -\delta \tilde{A}.$$

It exists thanks to (1.3). Hence up to replace $\tilde{A}$ by $\tilde{A} + df$ we can assume that $\delta \tilde{A} = 0$.

Finally, we set $A^0 = 0\, dx^0 + \tilde{A}_i\, dx^i$ and $A^1 = 0\, dx^0 + (E^0)_i\, dx^i$ and we consider a solution of (1.4). Hence $(1.5)_1$ is automatically satisfied. The fact that $A_{0,0|x_0=0} = A^1_0 = 0$ insures that we have

$$(\delta A)_{|x_0=0} = \delta \tilde{A} = 0,$$

which insures $(1.5)_2$. Then, using notably the fact that A_0 satisfies the wave equation and that $A_{0|x_0=0} = 0$, we get

$$\begin{aligned} (\delta A)_{,0})_{|x_0=0} &= (A_{0,00} - A_{i,i0})_{|x_0=0} \\ &= (-A_{i,0i})_{|x_0=0} \\ &= (-A^1)_{i,i|x_0=0} = \mathrm{div}(-E^0) = 0, \end{aligned} \tag{1.6}$$

which insures $(1.5)_3$. Finally

$$\begin{aligned} (dA)_{|x_0=0} &= (d(\tilde{A}) + A_{i,0}\, dx^0 \wedge dx^i)_{|x_0=0} \\ &= (\star_3 B^0 + E^0_i dx^0 \wedge dx^i)_{|x_0=0} \\ &= F_0 \end{aligned} \tag{1.7}$$

which insures $(1.5)_4$.

Claim 2: Taking A given by claim 1 then $F = dA$ is a solution of (ME)

proof of Claim 2:

It suffices to show that $\delta A = 0$ on $\mathbb{R}^4$. In this case we will have $\delta F = \delta dA = \delta dA + d\delta A = \Box A = 0$, $dF = 0$ is automatically satisfies and the initial value is a direct consequence of $(1.5)_4$.
But we eaily check that $\Box(\delta A) = \delta(\Box A) = 0$ and $(1.5)_2$ and $(1.5)_3$ imply that $(\delta A)_{|x_0=0} = ((\delta A)_{,0})_{|x_0=0} = 0$, by the uniquenness theorem for wave equation, see theorem 5 of §2.4 of [10], then $\delta A = 0$ on $\mathbb{R}^4$, which achieved the proof. $\Box$

Exercise 1.8 Show the solution of theorem 1.2.2 is unique.

We can notice that the constraint equation are easily solve in the vacuum case. If there is some charge there become

$$\begin{cases} \mathrm{div}(B^0) = 0 \\ \mathrm{div}(E^0) = \rho \end{cases}, \tag{CE}$$

Exercise 1.9 What condition ρ must satisfy in order that (CE) is solvable.

A variational approach

Let us consider the most natural functional on F, namely

$$\mathcal{L} : \{F \in \Omega^2(\mathbb{R}^{3,1}) \,|\, dF = 0\} \to \mathbb{R}$$

defined by

$$\mathcal{L}(F) = \frac{1}{4}\int_{\mathbb{R}^{3,1}} \langle F, F\rangle \, dv = \frac{1}{4}\int_{\mathbb{R}^{3,1}} F \wedge *F. \tag{1.8}$$

Using the electric and magnetic field it can we rewrite as

$$\mathcal{L}(F) = \frac{1}{2}\int_{\mathbb{R}^4} |B|^2 - |E|^2 \, dx.$$

Let us compute the Euler-Lagrange equation associated to this functional. Let $\alpha \in \Omega^2_c(\mathbb{R}^{3,1})$ such that $d\alpha = 0$, then there exists $\beta \in \Omega^1_c(\mathbb{R}^{3,1})$ such that $\alpha = d\beta$. Then

$$\mathcal{L}(F + t\alpha) = \mathcal{L}(F) + \frac{t}{2}\int_{\mathbb{R}^{3,1}} \langle F, d\beta\rangle \, dv + \ldots,$$

hence, integrating by part,

$$\frac{d\mathcal{L}(F + t\alpha)}{dt}_{|t=0} = \int_{\mathbb{R}^{3,1}} \langle \delta F, \beta\rangle \, dv,$$

which implies that

$$\delta F = 0. \tag{1.9}$$

Which corresponds exactly to the Maxwell's equation in the vacuum case.

The Energy-momentum tensor

In the previous paragraph, during the derivation of the Euler-Lagrange equation we have integrated by part throwing away the divergence terms. In this section we are going to see how Noether's Theorem permits to take advantage of this *a priori* no-effect divergence term. Here we will give a brief sake of the power of this theorem but we strongly invite the reader to have a look to the excellent review [16].

Let $\Omega \subset \mathbb{R}^s$ and M a sub-manifold of $\mathbb{R}^m$,

$$L : \{(x, p, q) \,|\, x \in \Omega, p \in M, q \in T_pM \otimes T^*_x\Omega\} \to \mathbb{R}$$

a C^1 function, then

$$\mathcal{L}(u) \int_\Omega L(x, u(x), du(x)) \, dx,$$

defines a functional over $C^1(\Omega, M)$.

Definition 1.2.3 *Let Ω, M and $\mathcal{L}$ as above. Let X a Lipschitz vector field on M. X is said an infinitesimal symmetry of $\mathcal{L}$ if*

$$\mathcal{L}(\phi_t^X(u)) = \mathcal{L}(u) + o(t) \text{ for all } u \in C^1(\Omega, M)$$

where ϕ_t^X is the flow defined by X.

The main idea of Noether is that any infinitesimal symmetry give rise to some divergence free quantity, i.e. some conserved quantity.

Theorem 1.2.4 (First Noether's Theorem, see theorem 3.1 of [14]) *Let Ω, M and $\mathcal{L}$ as above, and X an infinitesimal symmetry of $\mathcal{L}$, then if u is a critical point of $\mathcal{L}$ then*

$$\sum_{\alpha=1}^{s} \frac{\partial}{\partial x^\alpha}\left(X^j \frac{\partial L}{\partial q_\alpha^j}(x, u, du)\right) = 0,$$

or equivalently, the vector field

$$Y^\alpha = X^j \frac{\partial L}{\partial q_\alpha^j}(x, u, du)$$

is divergence free.

Exercise 1.10 Harmonic maps into the sphere

Let $S = \{u \in W^{1,2}(\mathbb{D}, \mathbb{R}^n) \mid |u| = 1\}$, we consider on S the classical Dirichlet energy

$$E(u) = \frac{1}{2}\int_{\mathbb{D}} |\nabla u|^2 \, dx.$$

1. Write the Euler-Lagrange equation.
2. Using the Invariance by the action of $SO(n)$, rewrite the Euler-Lagrange equation in divergence form.

The action defined in (1.8) is clearly invariant under the action of isometries of $\mathbb{R}^{3,1}$, which can be decomposed as a translation and a Lorentz transformation, i.e. an element of $O(3,1)$.However, here we are in a bit different situation than the one of the previous theorem since we act on the domain rather on the target.

More precisely, let $\psi : \mathbb{R}^{3,1} \to \mathbb{R}^{3,1}$ any positive isometry, we have

$$\mathcal{L}_\Omega(F) = \int_\Omega \|F(y)\|^2 \, dy = \int_{\psi^{-1}(\Omega)} \|(\psi^*(F))(x)\|^2 \, dx = \mathcal{L}_{\psi^{-1}(\Omega)}(\psi^* F).$$

Let us make the change of variable $x = \psi^{-1}(y)$ on the right-hand side

$$\mathcal{L}_{\psi^{-1}(\Omega)}(\psi^*(F)) = \int_\Omega \|(\psi^*(F))(\psi^{-1}(y)\|^2 \det(d\psi^{-1}(y))\, dy = \int_\Omega \|F(y)\|^2\, dy = \mathcal{L}_\Omega(F).$$

Since it is true for any Ω, we have finally have

$$\|(\psi^*(F))(\psi^{-1}(y)\|^2 \det(d\psi^{-1}(y)) = \|F(y)\|^2. \tag{1.10}$$

Let now assume that ψ is a deformation of identity, i.e. $\psi(x) = x + \varepsilon(x)$. Then we have

$$\begin{aligned} \psi^*(F) &= F_{kl}(\psi) d\psi^k \wedge d\psi^l = F_{kl}(\psi)(dx^k + \varepsilon^k_{,m} dx^m) \wedge (dx^l + \varepsilon^l_{,n} dx^n) \\ &= F_{kl}(\psi)(dx^k \wedge dx^l + \varepsilon^k_{,m} dx^m \wedge dx^l + \varepsilon^l_{,n} dx^k \wedge dx^n) + o(\varepsilon), \end{aligned} \tag{1.11}$$

hence

$$\psi^*(F)_{ab} = F_{ab}(\psi) + F_{lb}(\psi)\varepsilon^l_{,a} + F_{al}(\psi)\varepsilon^l_{,b},$$

moreover

$$det(d\psi^{-1}) = 1 - \varepsilon^l_{,l} + o(\varepsilon).$$

Plugging everything in (1.10) and using that $F_{ab} = -F_{ba}$, we get

$$\begin{aligned} \|F\|^2 &= \|F_{ab} + F_{lb}\varepsilon^l_{,a} + F_{al}\varepsilon^l_{,b}\|^2 (1 - \varepsilon^l_{,l}) + o(\varepsilon) \\ &= \|F\|^2 + 4F_{ab}F_l{}^b \varepsilon^{l,a} - \|F\|^2 \varepsilon^l_{,l} + o(\varepsilon) \end{aligned} \tag{1.12}$$

Subtracting we get

$$4F_{ab}F_l{}^b \varepsilon^{l,a} - 2F_{ab}F^{ab}\varepsilon^l_{,l} = 0$$

and integrating by part, we get

$$\begin{aligned} 0 = 4\left(F_{ab}F_l{}^b \varepsilon^l\right)^{,a} &- 4F^{,a}_{ab}F_l{}^b \varepsilon^l - 4F_{ab}F_l{}^{b,a}\varepsilon^l \\ &- \left(\|F\|^2 \varepsilon^l\right)_{,l} + 2F_{ab,l}F^{ab}\varepsilon^l\,. \end{aligned} \tag{1.13}$$

It is important to note that here we have kept the divergence term. Assuming that F is a critical point it must satisfies (1.9), that is to say

$$F_{ab}{}^{,a} = 0, \tag{1.14}$$

we get

$$\begin{aligned} 0 = 4\left(F_{ab}F_l{}^b \varepsilon^l\right)^{,a} &- 4F_{ab}F_l{}^{b,a}\varepsilon^l \\ &- \left(\|F\|^2 \varepsilon^l\right)_{,l} + 2F_{ab,l}F^{ab}\varepsilon^l\,. \end{aligned} \tag{1.15}$$

Moreover, using the fact that $dF = 0$, we get

$$F_{lb,a} + F_{ba,l} + F_{al,b} = 0 \tag{1.16}$$

which implies that

$$2F^{ab}(2F_{lb,a} - F_{ab,l}) = 2F^{ab}(F_{bl,a} + F_{lb,a} + F_{ba,l}) = 2F^{ab}(F_{al,b} + F_{lb,a} + F_{ba,l}) = 0 \tag{1.17}$$

Finally plugging (1.17) into (1.13), we get that

$$\left(4F_{ab}F_l{}^b\varepsilon^l - \|F\|^2\varepsilon_a\right)^{,a} = 0.$$

Which is true for any infinitesimal symmetry. We apply the previous formula with $\varepsilon^i(x) = c^i$, which corresponds to translation, then

$$T_{ab} = \frac{1}{4\pi}\left(F_{al}F_b{}^l - \frac{\eta_{ab}}{4}|F|^2\right)$$

is a symmetric divergence free tensor, called the Energy-Momentum tensor. We can easily check that

$$T_{00} = \frac{1}{8\pi}(|E|^2 + |B|^2)$$

and that

$$T_{00}^2 \geq \sum_{i=0}^{3} T_{0i}^2.$$

Hence the vector T_{0a} is timelike vector oriented to future. Which is interpreted as energy travel forward no faster than light.

Exercise 1.11 Applying the previous formula with $\varepsilon^i(x) = c^i_j x^j$ with $c^i_j = -c^i_i$, which correspond to infinitesimal rotation, show that

$$M^{abc} = x^a T^{bc} - x^c T^{ab}$$

is a free divergence quantity. It is called the Angular-Momentum Tensor.

Exercise 1.12 Considering the total energy at a given time, defined by

$$E(t)\int_{x_0=t} T_{00}\,dx.$$

Show that this quantity is conserved through time, i.e. $E'(t) = 0$.

To conclude this chapter, we would like to notice that this formulation of electromagnetism can be interpreted as $U(1)$- gauge (abelian) theory, see §3.5 and §4.1 of [5].

1.3 General Relativity and Einstein Equation

Let start with some reminder of Newtonian gravity. Let Ω be a relatively compact open subset of $\mathbb{R}^3$ and $\rho \in C_c^\infty(\Omega, \mathbb{R}^+)$ a density function which corresponds to our distribution of mass. Then the gravity potential must satisfy

$$\begin{cases} \Delta\Phi = 4\pi\rho, \\ \Phi(x) \to 0 \text{ when } |x| \to +\infty. \end{cases}$$

If the mass is punctual, we set $\rho = \delta_p$ and $\phi(x) = G(x,p) = -\frac{1}{4\pi|x-p|}$ is the Green function of $\mathbb{R}^3$ centered at p.

Then the gravity force is given by $\vec{F} = -\nabla\phi$ and the total mass is given by $m = \int_\Omega \rho\, dx$. We can compute ϕ thanks to Poisson formula,

$$\phi = G * \rho.$$

Since ρ is compactly supported it is not hard to see that

$$\phi(x) = \frac{c}{|x|} + O\left(\frac{1}{|x|^2}\right),$$

for some constant c, we are going to determined. Let $R > 0$ such that $\Omega \subset B(0,R)$ then

$$m = \int_{B(0,R)} \rho\, dx = \frac{1}{4\pi}\int_{B(0,R)} \Delta\phi\, dx = \frac{-1}{4\pi}\int_{\partial B(0,R)} \frac{\partial\phi}{\partial\nu}\, dx = c + O\left(\frac{1}{R}\right),$$

Finally

$$\phi(x) = \frac{m}{|x|} + O\left(\frac{1}{|x|^2}\right).$$

Hence the total mass can be measured by looking at the potential at infinity. But one main problem here is that the gravitation field spread instantaneously, which contradicts the fact that nothing (even information) can travel faster than light.

Postulate of General relativity

As underline previously, there is no gravitational force since it must propagate instantaneously. In fact the mass curved the space which explains the body fall. And this deformation of space travels through time as wave. A simplified image is the one of a tablecloth tensed on which we put some weighted balls which are going to deform the tablecloth.

Postulate of General relativity

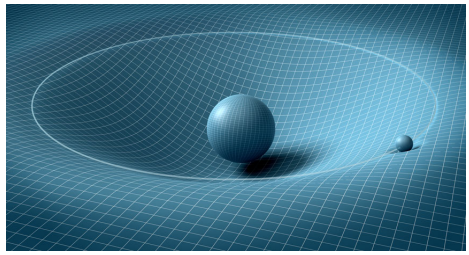

Figure 1.6: Space curved by the presence of mass

- The space-time is a 4-manifold (M, g) equipped with a Lorentzian metric.
- A particle at rest travel through geodesic of future type-time, i.e. solutions of

$$x''(t) + \Gamma^a_{bc} x'^b x'^c = 0,$$

  with $g(x', x') \leq 0$.
- g must satisfies the Einstein equations.

In the rest of the paragraph we are going to derive the Einstein equations. Einstein idea is that the gravity should propagate as a wave. Yet the Lagrangian associated to the wave equation is

$$\mathcal{L}(u) = \int_\Omega \left(-u_{,0}^2 + \sum_{i=1^3} u_{,i}^2 \right) dx,$$

which can be though as the Dirichlet functional associated to the Minkowsky metric η. In the Lorentzian case, we would like to build a Lagrangian which also involves only first derivatives. But it is impossible since no geometric information is contained in the first derivatives of the metric, since we can always assume their vanishes at a point, see §2.3 of [9]. Then we must use some curvature terms and since we want a scalar quantity, we naturally set

$$\mathcal{L}(g) = \int_M R \, dv_g,$$

where R is the scalar curvature and dv_g the volume form associated to g. It is known as the Einstein-Hilbert functional. Let us derive the Euler-Lagrange equation, for this we consider h a symmetric $(0, 2)$-tensor with compact support. Then

$$\begin{aligned} \delta_g(\mathcal{L}(g)) &\equiv \left. \frac{d\mathcal{L}(g + th)}{dt} \right|_{t=0} \\ &= \int_M \delta_g(g^{ab}) R_{ab} dv_g + g^{ab} \delta_g(R_{ab}) dv_g + g^{ab} R_{ab} \delta_g(dv_g) \end{aligned} \tag{1.18}$$

We get that

$$\delta_g(g^{ab}) = -g^{ac} h_{cd} g^{db}$$

and

$$\delta_g(dv_g) = \delta_g(\sqrt{-\det(g)})dx = \frac{1}{2}g^{ab}h_{ab}\sqrt{-\det(g)}dx = \frac{1}{2}\mathrm{tr}_g(h)dv_g. \tag{1.19}$$

In normal coordinates, thanks to (2.15) of [9], we have

$$R_{ab} = \Gamma^c_{ab,c} - \Gamma^c_{ca,b}$$

and thanks to proposition 2.5 of [9], see [24] chap 2 page §1.3 and §3.1 for the general case,

$$\delta_g(\Gamma^c_{ab}) = \frac{1}{2}\left(h^c_{b,a} + h^c_{a,b} - h^{,c}_{ab}\right)$$

which gives

$$\begin{aligned}\delta_g(R_{ab}) &= \frac{1}{2}\left(h^c_{b,a} + h^c_{a,b} - h^{,c}_{ab}\right)_{,c} - \frac{1}{2}\left(h^c_{a,c} + h^c_{c,a} - h^{,c}_{ca}\right)_{,b} \\ &= -\frac{1}{2}\Delta h_{ab} + \frac{1}{2}(h^c_{b,ac} + h^c_{a,bc} - h^c_{c,ab})\end{aligned} \tag{1.20}$$

then we exchange some derivatives which make appear some curvature thanks to [9],

$$\begin{aligned}\delta_g(R_{ab}) &= -\frac{1}{2}\Delta h_{ab} + \frac{1}{2}(h^c_{b,ca} + Rm^c{}_{lac}h^l_b - Rm^l{}_{bac}h^c_l + h^c_{a,cb} + Rm^c{}_{lcb}h^l_a - Rm^l{}_{acb}h^c_l - h^c_{c,ab}) \\ &= \frac{1}{2}\left(-\Delta h_{ab} + (h^c_{b,c})_{,a} + (h^c_{a,c})_{,b} - (h^c_c)_{,ab} - Ric_{la}h^l_b + Ric_{lb}h^l_a - Rm^l{}_{bac}h^c_l - Rm^l{}_{acb}h^c_l\right)\end{aligned} \tag{1.21}$$

Finally

$$g^{ab}\delta_g(R_{ab}) = -\Delta tr(h) + \mathrm{div}^2(h) = \mathrm{div}(-\nabla tr(h) + \mathrm{div}(h)) \tag{1.22}$$

Finally, since the last equation is a pure divergence term and using the fact h is compactly supported, then (1.18) becomes

$$\delta_g(\mathcal{L}(g)) = \int_M \left(-g^{ac}h_{cd}g^{db}R_{ab} + \frac{R}{2}\mathrm{tr}_g(h)\right) dv_g \tag{1.23}$$

Which leads to

$$-g^{ac}g^{db}R_{ab} + \frac{R}{2}g^{cd} = 0$$

that is to say, the Einstein equation in the vacuum case,

$$G \equiv Ric - \frac{R}{2}g = 0,$$

G is called the Eisntein tensor. Let remark that in dimension $n \neq 2$ this implies that $R = 0$, hence the Einstein equation reduces to $Ric = 0$.

Exercise 1.13

1. Using the second Bianchi identity proposition 2.14 of [9], show that G is divergence free.

2. (Hard) Show that G is divergence free using "only" the invariance by diffeomorphism of the Hilbert-Einstein functional.

In the General case the Lagrangian is given by

$$\mathcal{L}(g,\psi) = \int_M R + L_M(\psi, g)\, dv_g,$$

where ψ denotes the matter field and L_M the action of this matter field. Then, thanks to (1.19), we have,

$$\delta_g \left(L_M(\psi, g) dv_g\right) = \left(\frac{\partial L_M}{\partial g^{ab}} h^{ab} - \frac{g_{ab} h^{ab} L_M}{2}\right) dv_g.$$

Hence we define the stress energy tensor as

$$T_{ab} = \frac{1}{8\pi}\left(\frac{\partial L_M}{\partial g^{ab}} h^{ab} - \frac{g_{ab} L_M}{2}\right),$$

and the general Einstein equation are given by

$$G_{ab} = 8\pi T_{ab}.$$

We remark that, in dimension 4, $T \equiv g^{ab} T_{ab} = -\frac{1}{8\pi} R$.

One example example of field is given by the electromagnetic one, in this case

$$L_M(\psi, M) = \frac{1}{4}\langle \psi, \psi \rangle,$$

where ψ is a the two form associated to the electromagnetic field. See [31] for the scalar field or a perfect fluid field.

The dominant energy condition

With respect of the property of the Energy Momentum Tensor for the electromagnetic field in the Minkowsky space, it seems natural to make the following assumptions.

We consider an observer traveling in direction e^0, completed by e^i in order to have an orthonormal frame, then

- the observer observes a positive energy (or mass density), i.e.

$$T_{00} \geq 0.$$

- For any timelike future direction $\vec{e}$, $T_{ab} e^b$ which corresponds to the matter flow observe by someone traveling in the direction $\vec{e}$ must be of timelike with future

orientation, that is to say to the point of view of any observer nothing travel faster than light, which corresponds to

$$T_{00} \geq \sqrt{\sum_{i=1}^{3} T_{0i}^2}.$$

Static space-time

In this section we consider a simplified version of space-time in which time is ruled out.

Definition 1.3.1 *A space time (M^4, g) is called stationnary if there exits a Killing field K of timelike. Moreover if $K^\perp$ is integrable then we say that (M^4, g) is static.*

For a static space time, we choose a maximal slice (of spacelike) tangent to $K^\perp$, then we have $M = \mathbb{R} \times N$ with

$$g = -V^2\, dt^2 + h,$$

where h is a Riemannian metric on N, $K = \frac{\partial}{\partial t}$ and $V = V(x_1, x_2, x_3)$. Then we fix a Lorentzian frame with $e_0 = \frac{1}{V}\frac{\partial}{\partial t}$ and e_1, e_2 and e_3 a Riemannian frame of N.

Exercise 1.14

1. Compute the Christoffel symbol of $g = -V^2\, dt^2 + h$.
2. Then, using (2.15) of [9], show that the Ricci tensor of static manifold as above satisfies,

$$\begin{aligned} {Ric^M}_{00} &= \frac{\Delta_g V}{V}, \\ {Ric^M}_{0i} &= 0 \text{ for all } i \geq 1, \\ {Ric^M}_{ij} &= {Ric^N}_{ij} - \frac{\nabla_i \nabla_j V}{V} \text{ for all } i, j \geq 1. \end{aligned} \tag{1.24}$$

Hence we have the Einstein equation,

$$Ric^M = 8\pi \left(T_{ab} - \frac{T}{2} g_{ab} \right),$$

but in the static case we also assume that

$$T_{ab} = \begin{pmatrix} \rho & 0 \\ 0 & \tau_{ij} \end{pmatrix},$$

where ρ is the mass density and τ_{ij} the stress tensor. Hence we have

$$\begin{cases} \Delta_g V = 4\pi(\rho+\tau)V \\ VRic^N_{ij} - \nabla_i\nabla_j V = 4\pi(\tau_{ij} + (\rho-\tau)h_{ij}) \end{cases},$$

where $\tau = h^{ij}\tau_{ij}$.
In the vacuum case, $\rho = \tau_{ij} = 0$, we get

$$\begin{cases} \Delta_h V = 0 \\ VRic^N_{ij} - \nabla_i\nabla_j V = 0 \end{cases}$$

Taking the trace of the second equation, we have $VR^N = \Delta_h V = 0$, hence the scalar curvature must vanish, i.e.

$$R^N = 0 \tag{1.25}$$

One trivial solution of the Einstein equation is the static vacuum case is given by the Minkowsky space, where $N = \mathbb{R}^3$ with the euclidean metric and $V = 1$. In the next paragraph we derive a non trivial solution.

The Schwarzchild solution

Here we look to a non trivial solution of the Einstein equation, of course we are not able to solve the equation in a totally general framework, but considering radial solution, our PDEs will become ODEs and will permit us to solve it.

Theorem 1.3.2 *Every radial (static) solution of the Einstein equation is locally isometric to*

$$g_m = \left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{\rho}\right)^{-1} d\rho^2 + \rho^2 d\xi^2,$$

where $m \in \mathbb{R}$ and $d\xi^2$ is the standard metric on S^2.

Proof of theorem 1.3.2:

Then we try to solve (1.25). First of all since we look for a solution which is invariant under rotation, the metric on each sphere is proportional to the one of the standard sphere. Hence

$$h = u^4\delta,$$

for some radial function $u : \mathbb{R}^3 \to \mathbb{R}$ where δ is the euclidean metric. Thanks to (6.1) of [9] we know that u must satisfies

$$\Delta u = 0,$$

which implies that $u = a + \frac{b}{r}$. If a or b is zero, it corresponds to the flat case. Hence up to a dilation let us assume that

$$u = 1 + \frac{m}{2r},$$

then

$$h = \left(1 + \frac{m}{2r}\right)^4 (dr^2 + r^2 d\xi^2).$$

In order to find V now, we have to solve

$$\Delta_h V = 0.$$

Let set $\rho = r\left(1 + \frac{m}{2r}\right)^2$ then, in these new coordinates, we have

$$\begin{aligned} h &= \left(1 - \frac{2m}{\rho}\right)^{-1} d\rho^2 + \rho^2 d\xi^2 \\ &= f(\rho)^{-1} d\rho^2 + \rho^2 d\xi^2, \end{aligned} \tag{1.26}$$

where $f(\rho) = 1 - \frac{2m}{\rho}$. The equation becomes

$$\frac{1}{f^{-\frac{1}{2}}\rho^2}(\rho^2 f^{\frac{1}{2}} V')' = 0$$

which leads to

$$\rho^2 f^{\frac{1}{2}} V' = a$$

then

$$V' = a\left(1 - \frac{2m}{\rho}\right)^{-\frac{1}{2}} \frac{1}{\rho^2}$$

and finally

$$V = \tilde{a}\left(1 - \frac{2m}{\rho}\right)^{\frac{1}{2}}.$$

Normalizing V, it has the desired form. To conclude that this metric is a solution of the whole Einstein equation in vacuum we have to check that $V Ric^N_{ij} - \nabla_i \nabla_j V = 0$. □

We can remark that *a priori* g_m is only defined on $\mathbb{R} \times \mathbb{R}^3 \setminus B_{2m}$ and converges as infinity to the Minkowsky metric. For this last reason it is said asymptotically flat. On $\mathbb{R} \times \mathbb{R}^3 \setminus B_R$, with $R > 2m$, it can be considered as the solution of gravity outside a round star of radius bigger than R, where the vacuum condition is satisfied.
But as shown by the classical expression of the euclidean metric on the plane in polar coordinates, namely $dr^2 + rd\theta^2$, a singularity in the metric can be due to a bad choice of coordinates. A less trivial example consists in considering the metric $\frac{1}{t^4}dt^2 + dx^2$ a priori defined only for $t > 0$ but setting $t' = \frac{1}{t}$ it becomes $dt'^2 + dx^2$ which is nothing else than the metric of the cylinder an can be extended easily above $t = +\infty$ by considering $t' < 0$. A singularity in coordinates doesn't mean that the metric is really singular. In fact there is an easy way to check if at a point we have a true singularuity by considering the curvature. Indeed if the curvature blow we are sure that we have a true singularity. The converse is more delicate. When the curvature is bounded it can be sometime extended through the singularity and some time not. It is indeed the case for g_m, as shown in detail

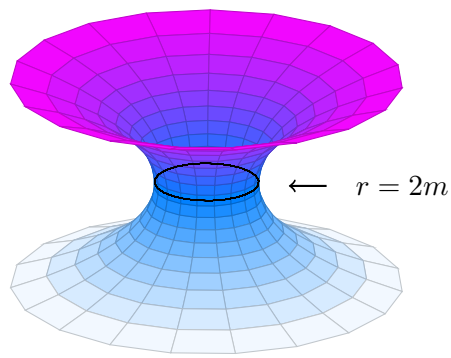

Figure 1.7: The spacial geometry of the Schwarzschild metric $\left(1+\frac{m}{2r}\right)^4\delta$.

in section 6.4 of [31], the metric can be extended up to the origin. However the origin is a true singularity. In fact, in a space-slice $t = c$, in the interior is of $B_{2m} \setminus \{0\}$ is the pullback of the metric through the inversion with respect to S_{2m}.

Here is the following change of coordinates that must be done. First setting

$$\begin{cases} u = t - r_* \\ v = t + r_* \end{cases}$$

where $r_* = r + 2m\ln\left(\frac{r}{2m} - 1\right)$, we get

$$g_m = -(1 - \frac{2m}{r})dudv.$$

Which can be rewrite

$$g_m = -\frac{2me^{-\frac{m}{2m}}}{r}e^{\frac{v-u}{4m}}dudv.$$

This suggest to set

$$\begin{cases} U = e^{-\frac{u}{4m}} \\ V = e^{\frac{v}{4m}} \end{cases}$$

which gives

$$g_m = -\frac{32m^3e^{-\frac{m}{2m}}}{r}dUdV.$$

Finally we set

$$\begin{cases} T = \frac{U+V}{2} \\ X = \frac{V-U}{2} \end{cases}$$

which gives a kind of weighted Minkowsky metric

$$g_m = -\frac{32m^3e^{-\frac{m}{2m}}}{r}(-dT^2 + dX^2) + r^2d\xi^2.$$

The relation between old and new coordinates is given by

$$\begin{cases} X^2 - T^2 = \left(\frac{r}{2m} - 1\right) e^{\frac{r}{2m}} \\ 2\tanh^{-1}\left(\frac{T}{X}\right) = \frac{t}{2m} \end{cases}$$

which permits us to draw the following Penrose diagram,

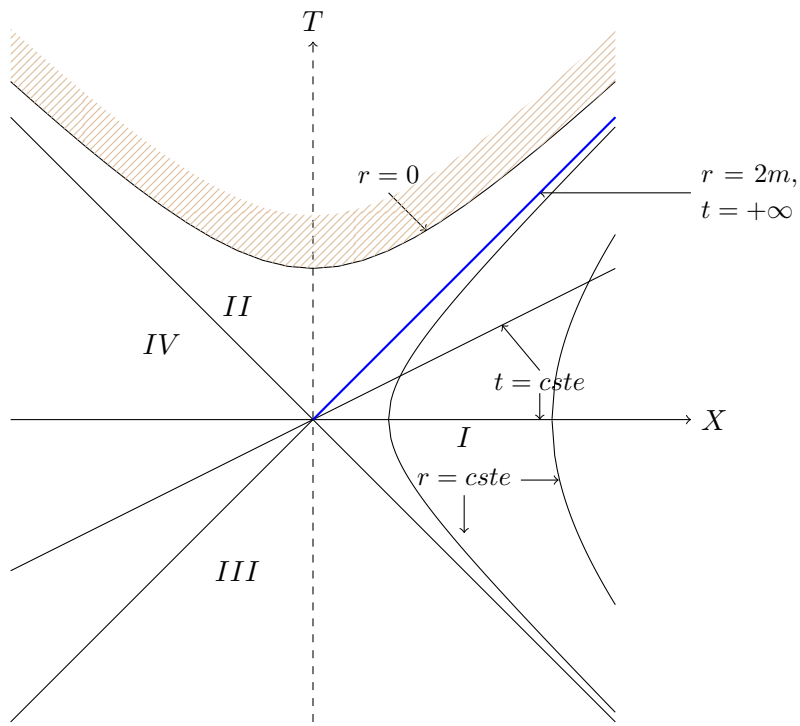

Figure 1.8: Penrose Diagram of the extension of the Schwarschild metric

However the time is still singular at $r = 2m$ and as show by the diagram, we see that no timelike geodesic can cross the sphere $r = 2m$, this sphere is called the event horizon. Since there is no constrained on the density of star, its radius can be as small as we want, in particular smaller than $2m$. In this case the star is inside the event horizon, it can't be seen by any observer at the exterior, we have a black hole. Of course the structure is not exactly this one up to r=0 since the black hole is not vaccum, see chapter 11 of [12] for a more precise picture.

To conclude this chapter we would like to notice the hypothesis of being static in theorem1.3.2 is surperfluous as shown by the Birkhoff theorem which insures that any rotationnally symmetric solution is static, see [13].

Chapter 2

Cauchy problem and Constraint equations

2.1 Origin of the constraint equations

We have seen in section 1.2, that we can solve the Maxwell equation if and only if the initial data satisfy some constraint equations. The Einstein equation being some generalization of the Maxwell equation replacing the electromagnetic field by the metric, it can be solved only for some particular initial data.

Let $(M, \tilde{g})$ be Lorentzian manifold satisfying the Einstein equation (**??**). Let $N \subset M$ a spacelike hypersurface with a future timelike unitary normal vector field e_0. Locally e_0 define a coordinate t such that $\frac{\partial}{\partial t} = e_0$. Then M is locally around N diffeomorphic to $N \times (-\varepsilon, \varepsilon)$. Then we obtain a family of metric $g_t = \tilde{g}_{|N_t}$ where $N_t = N \times \{t\}$. Hence we can consider $h = \frac{1}{2}\frac{dg_t}{dt}\Big|_{t=0}$, h is the second fundamental form of N with respect to e_0, see chapter 4 of [22] for some precise definitions, that is to say

$$(\widetilde{\nabla}_X Y)^{\perp} = h(X, Y)e_0$$

or equivalently

$$h(X, Y) = -\tilde{g}(\widetilde{\nabla}_X Y, e_0).$$

We can then reformulate our problem as follows: Given a triple (N, g, h), we look for $(M, \tilde{g})$ such that N is spacelike hypersurface of M and

$$\begin{cases} \widetilde{G}_{ab} = 8\pi T_{ab} \text{ on } M \\ \tilde{g}_{|N} = g \text{ on } N \\ \frac{1}{2}\frac{dg_t}{dt}_{|t=0} = h \text{ on } N \end{cases}$$

where the derivative is compute with respect to future timelike unitary normal vector to M.

Definition 2.1.1 *A triple* (M, g, h) *and* (ρ, J)[1] *is an initial data set if*

- M *is a complete* 3 *dimensional manifold,*
- g *is a Riemann metric on* M*,*
- h *is a symmetric* $(0,2)$*-tensor on* M*,*
- *It satisfies the constraint equation*

$$\begin{cases} R + (tr_g h)^2 - \|h\|_g^2 = 16\pi\rho \text{ on } M \\ div(h - (tr_g h)g) = 8\pi J \text{ on } M \end{cases} \tag{CE}$$

- *and finally the energy dominant condition*

$$\rho \geq \|J\|_g.$$

We say that (M, g, h) is vacuum if $\rho = J = 0$, in this case the constraint equations become

$$\begin{cases} R + (tr_g h)^2 - \|h\|_g^2 = 0 \text{ on } M \\ div(h - (tr_g h)g) = 0 \text{ on } M \end{cases} \tag{CEV}$$

We say that (M, g, h) is symmetric if $h = 0$. In this case the constraint equations force $J = 0$ and $R = 8\pi\rho \geq 0$. A weaker assumption often used, is that (M, g, h) is maximal, namely $tr_g h = 0$, which also implies that $R \geq 0$.

Where come from the constraint equations ? Let (e_1, e_2, e_3) an othonormal frame of (N, g) and e_0 a unit normal vector of N into M. Then we must satisfy the Gauss and Codazzy equation, see page 100 and 115 of [22],

$$\begin{cases} \tilde{R}m_{ijkl} = Rm_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk} \\ \nabla_j h^i_k - \nabla_k h^i_j = \tilde{R}m^i{}_{0jk} \end{cases} \tag{GCE}$$

Exercise 2.1 Let (N, g) a submanifold of $(M, \tilde{g})$. For any vector smooth field X and Y on M we consider some smooth extensions $\tilde{X}$ and $\tilde{Y}$ to M, and we set

$$II(X, Y) = \tilde{\nabla}_{\tilde{X}}\tilde{Y} - \nabla_X Y.$$

1. Show that II is well defined
2. Show that

   $$g(Rm(V, W)X, Y) = g(\tilde{R}m(V, W)X, Y) + \tilde{g}(II(Y, X), II(W, Y)) - \tilde{g}(II(V, Y), II(W, X)),$$

   for all X, Y, V and W tangent vector to N.

[1] ρ is the energy density and J the momentum vector.

3. Show that
$$(\tilde{R}m(V,W)X)^{\perp} = -\nabla_V II(W,X) + \nabla II_W(V,X)),$$
for all X, V and W tangent vector to N.

Taking twice the trace of the Gauss equation, we obtain

$$\begin{aligned} R + (tr_g h)^2 - \|h\|_g^2 &= \sum_{i,j=1}^{3} \tilde{R}m^{ij}{}_{ij} = \sum_{i,j=1}^{3} \tilde{R}m^{ij}{}_{ij} - \sum_{j=1}^{3} \tilde{R}m^{0j}{}_{0j} + \sum_{j=1}^{3} \tilde{R}m^{0j}{}_{0j} \\ &= \sum_{j=1}^{3} \tilde{R}ic_j^j - \tilde{R}ic_0^0 \\ &= \tilde{R} - 2\tilde{R}ic_0^0 = 2\tilde{G}_{00} = 16\pi T_{00} = 16\pi\rho \end{aligned} \tag{2.1}$$

To obtain, the second constraint we take the trace of the Codazzi equation, which gives

$$\nabla_i(h_k^i - tr_g(h)\delta_k^i) = \tilde{R}ic_{0k} = \tilde{G}_{0k} = 8\pi T_{0k} = 8\pi J_k.$$

Those consition are necessary and also sufficient thanks to the following theorem, see §16 of [25] for a proof.

Theorem 2.1.2 (Choquet-Bruhat and Géroch 69) *Let (N, g, h) an initial data set, then there exists a smooth manifold $(M, \tilde{g})$, called a maximal Cauchy development, satisfying:*

- *$(M, \tilde{g})$ satisfies the Einstein equations,*
- *(N, g) is an hypersurface of $(M, \tilde{g})$ of spacelike*
- *$\frac{dg_t}{dt}|_{t=0} = 2h$, where the derivative is computed in a future normal unitary direction of M,*
- *$(M, \tilde{g})$ is the maximal Lorentzian manifold to get those properties in the sens of inclusion.*

2.2 Resolution of the constraint equations in the vacuum case

In this section we are going to investigate what are the possible initial data set in a basic situation. This question in general setting is really hard and still open. It is nevertheless very interesting since it can be reformulated as: Which are the possible universes?
In the vacuum symmetric case, the constraint equations reduce to

$$R = 0.$$

We are going to find some metric with vanishing scalar curvature using **the conformal method**.

We fix a n-manifold (N, g_0), with $n \geq 3$, and we look for $u \in C^\infty(N)$ positive such that

$$R_g = 0 \text{ for } g = u^{\frac{4}{n-2}} g_0.$$

But (6.1) of [9] insures that the two scalar curvature are link through

$$\Delta_{g_0} u + \frac{n-2}{4(n-1)} R_{g_0} u = \frac{n-2}{4(n-1)} R_g.$$

Hence it suffices to find a solution of

$$L_{g_0}(u) = 0,$$

where $L_{g_0}(u) = \Delta_{g_0} u + \frac{n-2}{4(n-1)} R_{g_0} u$ is the conformal Laplacian.

Exercise 2.2 Show that for every positive $\phi \in C^\infty(M)$ and any $u \in C^\infty(M)$ we have

$$L_{g_\phi}(u) = \phi^{-\frac{n+2}{n-2}} L_{g_0}(\phi u)$$

where $g_\phi = \phi^{\frac{4}{n-2}} g_0$.

Proposition 2.2.1 *Let (N, g) be a compact smooth manifold such that $R_g \geq 0$ then for all $\tilde{g} \in [g]$[2], we get*

$$\int_N R_{\tilde{g}} \, dv_{\tilde{g}} \geq 0,$$

with equality only if $R_g \equiv 0$.

Proof:
Let $\tilde{g} = u^{\frac{4}{n-2}} g$, then

$$\Delta_g u + \frac{n-2}{4(n-1)} R_g u = \frac{n-2}{4(n-1)} u^{\frac{n+2}{n-2}} R_{\tilde{g}}$$

and

$$dv_{\tilde{g}} = \sqrt{|\tilde{g}|} = u^{\frac{2n}{n-2}} dv_g.$$

Hence, integrating the Yamabe equation against $u dv_g$, we get

$$\int_N |\nabla u|^2 \, dv_g + \frac{n-2}{4(n-1)} \int_N R_g u^2 \, dv_g = \int_N R_{\tilde{g}} \, dv_{\tilde{g}},$$

which implies the result, and in case of equality we must have $u \equiv cste$ which implies $R_g \equiv 0$. $\square$

[2] $[g] = \{u^{\frac{4}{n-2}} g \mid u \in C^\infty(N) \text{ and } u > 0\}$.

Corollary 2.2.2 *(N, g) be a compact smooth manifold such that $R_g \geq 0$ and $R_g \not\equiv 0$ then there is no conformal metric to g such that $R_{\tilde{g}} \equiv 0$.*

So there is no scalar flat metric on S^3 which is conformal to the standard one. However, this proposition doesn't lead to some topological obstruction, since we have the following result

Theorem 2.2.3 (Aubin, see § 3of [2]) *Let N a smooth compact n-manifold with $n \geq 3$, then there exists a metric g such that*

$$\int_N R_g \, dv_g < 0.$$

Exercise 2.3 Show that a consequence of the last result is that any manifold which admits a metric with positive total scalar curvature possess a scalar flat metric.

The stereographic projection:

Let $\pi : S^n \setminus \{N\} \to \mathbb{R}^n$ defined by

$$\pi(x) = \frac{1}{1 - x_{n+1}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

where $N = (0, \ldots, 0, 1)$.

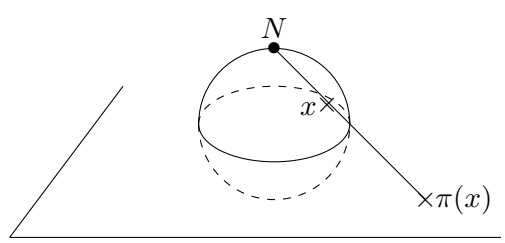

Figure 2.1: Stereographic projection

π is a conformal map, that is to say

$$d\pi \otimes d\pi = \lambda Id$$

for some smooth function λ.

Exercise 2.4

1. Show that

$$\pi^{-1}(y) = \frac{1}{1+|y|^2}\begin{pmatrix} y_1 \\ \vdots \\ y_n \\ |y|^2 - 1 \end{pmatrix}.$$

2. and that $d\pi^{-1} \otimes d\pi^{-1} = \frac{4}{(1+|y|^2)^2}$.

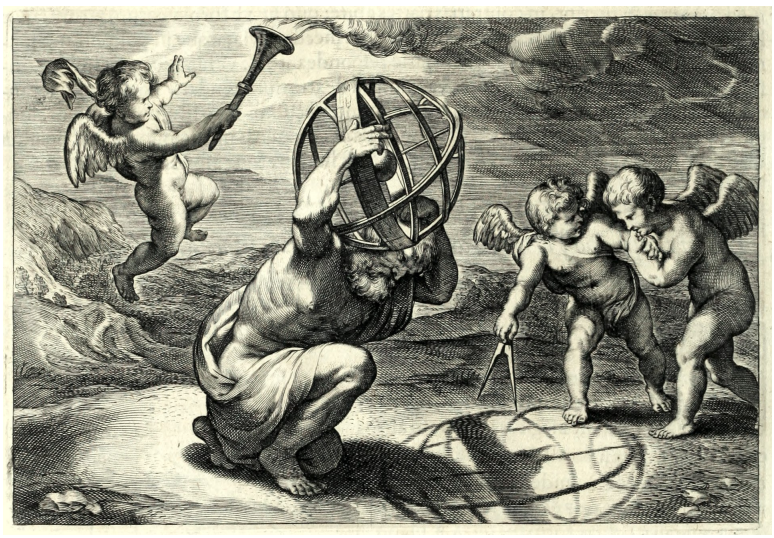

Figure 2.2: Rubens point of view on the projection

If we pull back the flat metric on S^n, by setting

$$g = \pi^*(\delta).$$

Since π is conformal we have

$$g = u^{\frac{4}{n-2}} g_0$$

where g_0 is the standard metric of S^n and u is a positive function. Finally we have

$$R_g \equiv 0 \text{ on } S^n \setminus \{N\}.$$

Hence we can put a scalar flat metric on the sphere up to remove a point. This process can be generalized to any compact manifold with a positive Yamabe invariant as show by the following theorem.

Theorem 2.2.4 *Let (N, g) be a smooth compact manifold with $R_g \geq 0$ and $\int_N R_g \, dv_g > 0$* [3]*. Then there exists $G : N \times N \setminus D \to \mathbb{R}$, where $D = \{(x, x) \,|\, x \in N\}$, such that*

[3] In fact this hypothesis can be replace by $Y([g]) = \inf_{\tilde{g} \in [g]} \frac{1}{Vol_{\tilde{g}}^{\frac{n-2}{n}}} \int_N R_{\tilde{g}} \, dv_{\tilde{g}} > 0$

- *G is smooth, positive and symmetric on $N \times N \setminus D$,*
- *for all $x \in N$ we have $G(x, \,.\,) \in L^1(N)$ and*

$$\int_N G(x,y) L_g(\phi)(y)\, dv_g = \phi(x)$$

for all $\phi \in C^\infty(N)$.
G is called the Green function of the conformal Laplacian.

Proof of theorem 2.2.4:
We proceed in several steps. First we approximate the Green function of the Laplacian by some analogue with the euclidean case. Let $0 < \delta < \frac{inj(N)}{3}$ and $\eta \in C_c^\infty(N \times N, [0,1])$ symmetric such that

$$\begin{cases} \eta(x,y) = 1 \text{ if } d_g(x,y) \leq \delta \\ \eta(x,y) = 0 \text{ if } d_g(x,y) \geq 2\delta \end{cases}$$

We set

$$H(x,y) = \frac{\eta(x,y)}{(n-2)\omega_{n-1} d_g(x,y)^{n-2}}.$$

Step 1: for a any $\phi \in C^\infty(N)$ we get

$$\int_N \Delta_g \phi(y) H(x,y)\, dv_g = \phi(x) + \int_N \phi(y) \Delta_g H(x,y)\, dv_g. \tag{2.2}$$

Indeed, let $0 < \varepsilon < \frac{\delta}{2}$,

$$\begin{aligned} \int_N \Delta_g \phi(y) H(x,y)\, dv_g &= \int_{N \setminus B(x,\varepsilon)} \Delta_g \phi(y) H(x,y)\, dv_g + \int_{B(x,\varepsilon)} \frac{\Delta_g \phi(y)}{(n-2)\omega_{n-1} d_g(x,y)^{n-2}}\, dv_g \\ &= \int_{N \setminus B(x,\varepsilon)} \Delta_g \phi(y) H(x,y)\, dv_g + O(\varepsilon^2) \\ &= \int_{\partial B(x,\varepsilon)} -\partial_\nu \phi(y) H(x,y) + \phi(y) \partial_\nu H(x,y)\, dv_g \\ &+ \int_{N \setminus B(x,\varepsilon)} \phi(y) \Delta_g H(x,y)\, dv_g + O(\varepsilon^2) \\ &= \frac{1}{\varepsilon^{n-1}\omega_{n-1}} \int_{\partial B(x,\varepsilon)} \phi(y)\, dv_g + \int_{N \setminus B(x,\varepsilon)} \phi(y) \Delta_g H(x,y)\, dv_g + O(\varepsilon) \end{aligned} \tag{2.3}$$

Exercise 2.5

1. Show that in coordinates

$$\Delta_g(f) = \partial^j f \partial_j \ln(\sqrt{|g|}) + \partial_j (g^{ij} \partial_i f).$$

2. If f is radial, i.e. constant on spher e in normal coordinates, then show that, still in normal coordinates,
$$\nabla_g f = \lambda y.$$
and then, using this fact that for $f = \frac{\mu}{d_g(c,y)^{n-2}}$ then
$$\partial_j(g^{ij}\partial_i f) = 0.$$

3. Finally, in normal coordinates centererd at x, we get
$$\Delta_g H(x,y) = \Delta_g\left(\frac{1}{(n-2)\omega_{n-1}d_g(x,y)^{n-2}}\right) = \frac{1}{2\omega_{n-1}}|y|^{-n}y^i\partial_i(\ln(|g|) \tag{2.4}$$

Thanks to the previous exercise, since in normal coordinates $\partial_i|g|(0) = 0$, then we have
$$|\Delta_g H(x,y)| \leq \frac{C}{|y|^{n-2}}$$
hence
$$|\Delta_g H(x,y)| \in L^1.$$
Taking the limit as $\varepsilon \to 0$ we get the (2.2).

Now we solve the linear part through an iteration process.
Step 2: Setting
$$F_1 = -L_g H = -\Delta_g H(x,y) - \frac{n-2}{4(n-1)}R_g(x)H(x,y)$$
and
$$F_{n+1} = \int_N F_k(x,z)F_1(z,y)\,,dv_g.$$
Then for $k > \frac{n}{2}$, F_{k+1} is continuous.

Exercise 2.6 Show that if $F, G : N \times N \setminus D \to \mathbb{R}$ are two continuous functions such that, there exist $\alpha, \beta > 0$ and $C_F, C_G > 0$, such that
$$|F(x,y)| \leq \frac{C_F}{d_g(x,y)^{n-\alpha}},\ G(x,y) \leq \frac{C_G}{d_g(x,y)^{n-\beta}}$$
then
$$\left|\int_N F(x,z)G(z,y)dv_g\right| \leq C_F C_G C(N,g,\alpha,\beta)\begin{cases} \frac{1}{d_g(x,y)^{n-\alpha-\beta}} & \text{if } \alpha+\beta < n, \\ 1 + |\ln(d_g(x,y)| & \text{if } \alpha+\beta = n, \\ 1 & \text{if } \alpha+\beta > n. \end{cases}$$

Using the previous exercise, we easily prove that F_k si bounded for $k > \frac{n}{2}$ and get the desired result. **Step 3: There exists $u_x \in W^{2,2}(N)$ such that**

$$L_g(u_x) = F_{k+1}(x, \,.\,).$$

In order to find u_x, let us minimize

$$\int_N uL_g(u)\, dv_g$$

under the constraint

$$\int_N F_{k+1}(x, \,.\,)u\, dv_g = 1.$$

We can assume that the set on which we minimize is not empty that is to say that $F_{k+1} \not\equiv 0$, else $u_x \equiv 0$ trivially solves the problem.

Exercise 2.7 Show that, since $R_g \geq 0$, the functional

$$u \mapsto \int_N uL_g(u)\, dv_g = \int_N \left(|\nabla u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) dv_g$$

is corecive on $W^{1,2}(N)$.

Let take a minimizing sequence u_n, it is bounded by coercivity, hence we can extract a subsequence, still denoted u_n, such that

$$u_n \rightharpoonup u \text{ in } W^{1,2}$$

and

$$u_n \to u \text{ in } L^2.$$

We easily check that u is a solution of the minimization problem, hence there exists $\mu \in \mathbb{R}$, a Lagrange multiplier, such that

$$L_g(u) = \mu F_{k+1}.$$

Moreover, we easily get that

$$\mu = \int_N uL_g(u)\, dv_g = \int_N \left(|\nabla u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) dv_g > 0.$$

Hence, $\frac{u}{\mu}$ solves our problem and the elliptic regularity permits us to conclude step 3.
Step 4:

$$G(x,y) = H(x,y) + \sum_{i=1}^{k} \int_N F_i(x,z) H(z,y)\, dv_g + u_x$$

is the Green function of the conformal Laplacian.
Indeed

$$\begin{aligned}
L_gG &= L_gH + \sum_{i=1}^{k}\int_N L_gF_i(x,z)H(z,y)\,dv_g + F_{k+1}\\
&= \delta_x - F_1 + \sum_{i=1}^{k}\int_N F_i(x,z)(\delta_x + \Delta_g H + \frac{n-2}{4(n-1)}R_gH)\,dv_g + F_{k+1}\\
&= \delta_x - F_1 + \sum_{i=1}^{k} F_i - \int_N F_i(x,z)F_1\,dv_g + F_{k+1}\\
&= \delta_x - F_1 + \sum_{i=1}^{k} F_i - F_{i+1} + F_{k+1}\\
&= \delta_x
\end{aligned} \tag{2.5}$$

Step 5: G is strictly positive
It is a direct consequence of the maximum principle. □

Theorem 2.2.5 *Let g a smooth metric on a compact manifold N, for any $x_0 \in N$,there exists $\tilde{g} \in [g]$ such that*

$$Ric(x_0) = 0 \tag{2.6}$$

and

$$sym(\nabla Ric)(x_0) = 0 \tag{2.7}$$

which notably implies

$$R_g(x_0) = O(|y|^2). \tag{2.8}$$

Corollary 2.2.6 *Let g a smooth metric on a compact manifold N, there exists $\tilde{g} \in [g]$ such that*

$$|\tilde{g}| = 1 + (|y|^4). \tag{2.9}$$

Proof of Corollary:

Let us remind, see proposition 2.15 of [9], that in normal coordinates, we have

$$g_{ij}(y) = \delta_{ij} + \frac{1}{3}Rm_{i\alpha\beta j}y^\alpha y^\beta + \frac{1}{6}Rm_{i\alpha\beta j,\gamma}y^\alpha y^\beta y^\gamma + O(|y|^4)$$

which implies that

$$\sqrt{|g|}(y) = 1 - \frac{1}{6}Ric_{\alpha\beta}y^\alpha y^\beta - \frac{1}{6}Ric_{(\alpha\beta,\gamma)}y^\alpha y^\beta y^\gamma + O(|y|^4),$$

hence the proof follow from the one of theorem 2.2.5. □

Proof of theorem 2.2.5:
We just need to find a conformal change of metric, $\tilde{g} = u^{\frac{4}{n-2}}$, such that

$$Ric_{\tilde{g}}(x_0) = \nabla Ric_{\tilde{g}}(x_0) = 0.$$

Thanks to (2.37) of [9], we get

$$Ric_{\tilde{g}_{\alpha\beta}}(x_0) = Ric_{g_{\alpha\beta}}(x_0) - \frac{n-2}{2} d^2u(x_0) + \frac{1}{2}\Delta u(x_0)\,\delta_{\alpha\beta} - \frac{n-2}{4}|\nabla u(x_0)|^2\delta_{\alpha\beta} + \frac{n-2}{4} du \otimes du(x_0),$$

and we deduce that, in normal coordinates, that

$$\begin{aligned} Ric_{\tilde{g}_{(\alpha\beta,\gamma)}}(x_0) = {} & Ric_{g_{(\alpha\beta,\gamma)}}(x_0) - \frac{3(n-2)}{2}\partial_{\alpha\beta\gamma}u(x_0) + \frac{n-2}{2}\Gamma^l_{(\alpha\beta,\gamma)}\partial_l u(x_0) \\ & + \frac{1}{2}(\partial^l_{\gamma ul}u(x_0)\,\delta_{\alpha\beta} + \partial^l_{\alpha ul}u(x_0)\,\delta_{\beta\gamma} + \partial^l_{\beta ul}u(x_0)\,\delta_{\gamma\alpha}) \\ & - \frac{n-2}{2}(\partial_{l\gamma}u(x_0)\,\delta_{\alpha\beta} + \partial_{l\alpha}u(x_0)\,\delta_{\beta\gamma} + \partial_{l\beta}u(x_0)\,\delta_{\gamma\alpha})\partial^l u \\ & + \frac{n-2}{2}(\partial_{\alpha\beta}u(x_0)\partial_\gamma u(x_0) + \partial_{\beta\gamma}u(x_0)\partial_\alpha u(x_0) + \partial_{\gamma\alpha}u(x_0)\partial_\beta u(x_0)). \end{aligned} \tag{2.10}$$

If we set

$$u = A_{ij}y^iy^j + B_{ijk}y^iy^jy^k$$

Then, (4.13) is equivalent to

$$Ric_{g_{\alpha\beta}}(x_0) - \frac{n-2}{2}(A_{\alpha\beta} + A_{\beta\alpha}) - A^l_l\delta_{\alpha\beta} = 0$$

then

$$A_{\alpha\beta} = \frac{1}{n-2}Ric_{g_{\alpha\beta}}(x_0) - \frac{1}{2(n-1)(n-2)}R_g(x_0),$$

solves (4.13).
Moreover setting

$$B_{\alpha\beta,\gamma} = \frac{1}{3(n-2)}Ric_{g_{\alpha\beta,\gamma}}(x_0) - \frac{1}{6(n-1)(n-2)}R_{g,\gamma}(x_0)\delta_{\alpha\beta}$$

and using the second Bianchi identity, we solve (2.7). □

Proposition 2.2.7 *Let (N, g) a compact Riemannian manifold of dimension $n \in [3, 5]$, assume that g verifies (2.8) and (2.9), then, we have*

$$G(x, y) = \frac{1}{(n-2)\omega_{n-1}d_g(c, y)^{n-2}} + O(1).$$

where G is the Green function of the conformal Laplacian.

Proof of proposition 2.2.7:
Since is the G the Green function of the conformal Laplacian, we have

$$\Delta_g G + \frac{n-4}{2(n-1)} R_g G = 0 \text{ on } M \setminus \{x\},$$

with, by construction,

$$G(x,y) = \frac{1}{(n-2)\omega_{n-1} d_g(x,y)^{n-2}} + \beta(x,y)$$

where $\beta(x,\,.\,) \in L^p$ for $p > 1$. Hence

$$\Delta_{\tilde{g}}\beta + \frac{n-4}{2(n-1)} R_{\tilde{g}}\beta = -\Delta_{\tilde{g}}\left(\frac{1}{(n-2)\omega_{n-1} d_g(x,y)^{n-2}}\right) - \frac{n-4}{2(n-1)}\left(\frac{R_{\tilde{g}}}{(n-2)\omega_{n-1} d_g(x,y)^{n-2}}\right).$$

Thanks to (2.9),(2.4), (2.8) and the fact that $3 \leq n \leq 5$, we have

$$\Delta_{\tilde{g}}\beta + \frac{n-4}{2(n-1)} R_{\tilde{g}}\beta = O\left(\frac{1}{d_g(x,y)}\right).$$

By standard elliptic theory $\beta \in W^{2,p}$ for every $p < n$. Finally Sobolev injections insure that $\beta \in L^\infty$. □

Finally, we can consider the analytical blow-up of a compact manifold (N, g) . let fix a point x, we first apply theorem 2.2.5 in order to get $\tilde{g} \in [g]$ which is sufficiently flat at X, then we make a new conformal change of metric by setting $\bar{g} = \left((n-2)\omega_{n-1}\tilde{G}\right)^{\frac{4}{n-2}} \tilde{g}$, where $\tilde{G}$ is the Green function of the conformal laplacian of $\tilde{g}$. Then the new manifold $(N \setminus \{x\}, \bar{g})$ is complete, scalar flat. Let us have look of the behavior of the metric when it approaches x.
Let us consider normal coordinate in N around x and perform an inversion by $z^i = \phi(y) = \frac{y^i}{|y|^2}$. Then, in new this coordinates,

$$\bar{g}_{\alpha\beta} = (\phi^*(\bar{g}))_{\alpha\beta} = |d\phi|^2 \bar{g}_{\alpha\beta} = \frac{1}{|y|^4}\bar{g}_{\alpha\beta} = \frac{1}{|y|^4}\left(|y|^{n-2} + O(1)\right)^{\frac{4}{n-2}} \tilde{g}_{\alpha\beta}.$$

In the last equality, we used proposition 2.2.7. Finally using the fact we initially chose normal coordinates, we get

$$\tilde{g}_{\alpha\beta} = \left(1 + O\left(|y|^{2-n}\right)\right)^{\frac{4}{n-2}} \left(\delta_{\alpha\beta} + O(|y|^{-2})\right)$$

Hence

$$\begin{aligned} \tilde{g} &= \delta_{\alpha\beta} + O\left(\tfrac{1}{|y|^2}\right) \quad \text{if } n \geq 4 \\ \tilde{g} &= \delta_{\alpha\beta} + O\left(\tfrac{1}{|y|}\right) \quad \text{if } n = 3. \end{aligned}$$

Finally we see that the analytic blow-up is an asymptotically flat.

Exercise 2.8 Since we used proposition 2.2.7, the previous claim is *a priori* only true for $n \geq 5$. Show that it is also true for $n \geq 6$. Indices, check lemma 6.4 of [18].

Exercise 2.9 [not very easy]

1. Show that the $(\mathbb{R}^3, \delta)$ is obtain blowing-up the standard 3-sphere.
2. Show that the Schwarzschild metric is obtain by a double cover of a blow-up of the projective space, i.e. $\mathbb{R}\mathrm{P}^3$.

Chapter 3

Asymptotically flat manifolds and mass

At the end of the previous chapter we have seen that we can build many scalar flat manifold whose metric become flat around the blow-up point. In this chapter we are going to focus on those type of manifold which are asymptotic to the euclidean infinity. We are going to consider not only the scalar flat one as in the previous chapter but all the one with a finite total curvature, i.e.

$$\int_N R_g \, dv_g < +\infty.$$

Contrary to the compact case, the study of elliptic PDE will require some additional control at infinity, especially on the decreasing rate of function.

3.1 Weighted spaces

To start we defined the notion of weighted spaces in $\mathbb{R}^n$, it will be enlarged to manifold later through some chart.

Definition 3.1.1 *On $\mathbb{R}^n$, let $\sigma = (1+r^2)^{\frac{1}{2}}$ and $\delta \in \mathbb{R}$, we set*

- $L^p_\delta = \left\{ u \in L^p_{loc} \mid \int_{\mathbb{R}^n} |u|^p \sigma^{-\delta p - n} \, dx < +\infty \right\}$, *we equip this set with the norm* $\|u\|^p_{p,\delta} = \int_{\mathbb{R}^n} |u|^p \sigma^{-\delta p - n} \, dx$.
- $L^\infty_\delta = \left\{ u \in L^\infty_{loc} \mid \sup_{\mathbb{R}^n} |u| \sigma^{-\delta} < +\infty \right\}$, *we equip this set with the norm* $\|u\|_{\infty,\delta} = \sup\limits_{\mathbb{R}^n} |u| \sigma^{-\delta}$.
- $L'^p_\delta = \left\{ u \in L^p_{loc}(\mathbb{R}^n \setminus \{0\}) \mid \int_{\mathbb{R}^n \setminus \{0\}} |u|^p r^{-\delta p - n} \, dx < +\infty \right\}$, *we equip this set with the norm* $\|u\|'^p_{p,\delta} = \int_{\mathbb{R}^n \setminus \{0\}} |u|^p r^{-\delta p - n} \, dx$.

- $L'^{\infty}_{\delta} = \left\{ u \in L^{\infty}_{loc}(\mathbb{R}^n \setminus \{0\}) \mid \sup_{\mathbb{R}^n \setminus \{0\}} |u| r^{-\delta} < +\infty \right\}$, *we equip this set with the norm* $\|u\|'_{\infty,\delta} = \sup\limits_{\mathbb{R}^n \setminus \{0\}} |u| r^{-\delta}$.

- $W^{k,p}_{\delta} = \left\{ u \in W^{k,p}_{loc} \mid \sum_{j=0}^{k} \|d^j u\|_{p,\delta-j} < +\infty \right\}$, *we equip this set with the norm* $\|u\|_{k,p,\delta} = \sum_{j=0}^{k} \|d^j u\|_{p,\delta-j}$.

- $W'^{k,p}_{\delta} = \left\{ u \in W^{k,p}_{loc}(\mathbb{R}^n \setminus \{0\} \mid \sum_{j=0}^{k} \|d^j u\|'_{p,\delta-j} < +\infty \right\}$, *we equip this set with the norm* $\|u\|'_{k,p,\delta} = \sum_{j=0}^{k} \|d^j u\|'_{p,\delta-j}$.

We immediately remark that:

a) C^{∞}_{c} is dense in $W^{k,p}_{\delta}$ if $p < +\infty$,

b) $L^{p}_{-\frac{n}{p}} = L^p$,

c) if $u \in L^{\infty}_{\delta}$ then $|u| = O(r^{\delta})$ at infinity.

d) let $R \geq 1$, there exists $C > 0$ depending only on n,p and δ, if $u_R(x) = u(Rx)$ then

$$\frac{1}{C} R^{-\delta} \|u_{R|A_1}\|_{k,p,\delta} \leq \|u_{|A_R}\|_{k,p,\delta} \leq C R^{-\delta} \|u_{R|A_1}\|_{k,p,\delta},$$

for $A_R = B_{2R} \setminus B_R$.

The next properties we would like to underline deserve a full theorem.

Theorem 3.1.2 *let* $1 \leq p \leq q \leq \infty$ *and* $\delta_2 < \delta_1$, *then*

1. *if* $u \in L^{q}_{\delta_2}$ *then*
$$\|u\| p, \delta_1 \leq \|u\| q, \delta_2,$$
which implies that $L^{q}_{\delta_2} \subset L^{p}_{\delta_1}$.

2. *Hölder inequality: if* $u \in L^{q}_{\delta_1}$ *and* $v \in L^{p}_{\delta_2}$, *with* $\delta = \delta_1 + \delta_2$ *and* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, *then*
$$\|uv\|_{r,\delta} \leq \|u\|_{q,\delta_1} \|v\|_{p,\delta_2}.$$

3. *Interpolation Inequality: for all* $\varepsilon > 0$ *there exists* $C(\varepsilon) > 0$ *such that for all* $u \in W^{2,p}_{\delta}$,
$$\|u\|_{1,p,\delta} \leq \varepsilon \|u\|_{2,p,\delta} + C(\varepsilon) \|u\|_{p,\delta}.$$

4. *Sobolev inequality: for* $u \in W^{k,p}_{\delta}$ *if* $n - kp > 0$ *then*
$$\|u\|_{\frac{np}{n-p},\delta} \leq C \|u\|_{k,p,\delta},$$

if $n - kp < 0$ *then*

$$\|u\|_{\infty,\delta} \leq C\|u\|_{k,p,\delta}$$

and moreover

$$|u| = o(r^{\delta}).$$

Proof of theorem :
We let 1,2 and 3 as exercises we will focus on the proof of 4 which exhibit the main technique. If $n - kp > 0$, let $p^* = \frac{np}{n-p}$ and $r \geq 1$, then

$$\|u_{|A_R}\|_{p^*,\delta} \leq CR^{-\delta}\|u_{R|A_1}\|_{p^*,\delta} \leq CR^{-\delta}\|u_{R|A_1}\|_{k,p,\delta},$$

where in the last inequality we used the classical Sobolev injection on A_1 and the fact σ is uniformly controlled above and below on A_1. Finally on each annulus A_R, we get

$$\|u_{|A_R}\|_{p^*,\delta} \leq C4\|u_{A_R}\|_{k,p,\delta}.$$

let $u_j = u_{|B_1}$ if $j = 0$ and $u_j = u_{|A_{2^{j-1}}}$ if $j \geq 1$, then

$$\begin{aligned} \|u\|_{p^*,\delta} &= \left(\sum_{j=0} \|u_j\|_{p^*,\delta}^{p^*}\right)^{p^*} \\ &\leq C\left(\sum_{j=0} \|u_j\|_{k,p,\delta}^{p^*}\right)^{p^*} \\ &\leq C\left(\sum_{j=0} \|u_j\|_{k,p,\delta}^{p}\right)^{p}, \end{aligned} \tag{3.1}$$

the last inequality is a consequence of the fact that $p^* \geq p \geq 1$. If $n - kp < 0$ we have smilingly

$$\sup_{A_R} |u|\sigma^{-\delta} \leq C\|u_{|A_R}\|_{k,p,\delta}$$

which gives also the desired injection. But considering the previous inequality when R become large we get

$$|u| = o(r^{\delta}),$$

which achieves the proof of 4. □ We remark that 1) is optimal by considering $u(r) = \frac{1}{\ln(r)}$ with $\delta_1 = \delta_2 = 0$, $p = 1$ and $q = 2$. We have, up to reduce the exponent, a similar Sobolev emebedding than the usual one

Theorem 3.1.3 *For* $k > j$ *and* $\delta < \varepsilon$ *the injection*

$$W_{\delta}^{k,p} \subset W_{\varepsilon}^{j,p}$$

is compact.

Exercise 3.1 Prove the previous theorem

Theorem 3.1.4 (Poincaré inequality) *Let $\delta < 0$ and $p \geq 1$, there exists $C > 0$, such that for any $u \in W^{1,p}_\delta$ we have*

$$\|u\|_{p,\delta} \leq C \left\| \frac{\partial u}{\partial r} \right\|_{p,\delta-1}.$$

Proof of theorem 3.1.4:
As C^∞_c is dense in $W^{1,p}_\delta$, we consider $u \in C^\infty_c$. Moreover, we easily proved that

$$\Delta(\sigma^{2-n}) = n(n-2)\sigma^{-2-n}.$$

Hence, by integration by parts, we get

$$\begin{aligned}\int_{\mathbb{R}^n} \nabla\sigma^{2-n}\nabla(\sigma^{-\delta p}|u|^p)\,dx &= n(n-2)\int_{\mathbb{R}^n} \sigma^{-2-n}\sigma^{-\delta p}|u|^p\,dx \\ &= (2-n)\int_{\mathbb{R}^n} \sigma^{-n} r(-\delta p\sigma^{-\delta p-2} r|u|^p + \sigma^{-\delta p} p\partial_r|u||u|^{p-1})\,dx\end{aligned} \tag{3.2}$$

which leads, setting $u_r = \frac{\partial u}{\partial r}$, to

$$\begin{aligned}&\int_{\mathbb{R}^n} (-(n-2)\delta p r^2 + n(n-2))\sigma^{-n-2-\delta p}|u|^p\,dx = (2-n)\int_{\mathbb{R}^n} pr\sigma^{-n-\delta p}\partial_r|u||u|^{p-1}\,dx \\ &\leq (n-2)p\left(\int_{\mathbb{R}^n} (\sigma^{-\delta+1-\frac{n}{p}})^p \left(\frac{r}{\sigma}\right)^p |u_r|^p \left(\lambda^{\frac{1-p}{p}}\right)^p dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \left(\sigma^{-n\frac{p-1}{p}-\delta(p-1)}|u|^{p-1}\lambda^{\frac{p-1}{p}}\right)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \\ &\leq (n-2)p\left(\int_{\mathbb{R}^n} \sigma^{p(-\delta+1)-n}\left(\frac{r}{\sigma}\right)^p |u_r|^p\lambda^{1-p}\,dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \sigma^{-n-\delta p}|u|^p\lambda\,dx\right)^{\frac{p-1}{p}}\end{aligned} \tag{3.3}$$

where

$$(n-2)p\lambda = \frac{-(n-2)\delta p r^2 + n(n-2)}{1+r^2}.$$

Since $\delta < 0$ then $\lambda > 0$, we get

$$\int_{\mathbb{R}^n} |u|^p\lambda\sigma^{-\delta p-n}\,dx \leq \int_{\mathbb{R}^n} |u_r|^p\lambda^{1-p}\left(\frac{r}{\sigma}\right)^p \sigma^{p(-\delta+1)-n}\,dx,$$

finally remarking that λ is continuous and that

$$\lambda \to |\delta| \text{ as } r \to +\infty$$

we get that λ is bounded above and below and proves the theorem. □

3.2 Elliptic operator asymptotic to the Laplacian

Definition 3.2.1 *A differential operator $u \mapsto P(u)$ defined by*

$$P(u) = a^{ij}\partial_{ij}u + b^i\partial_i u + cu$$

where $a^{ij}, b^i, c : \mathbb{R}^n \to \mathbb{R}$ are continuous, is said asymptotic to the Laplacian at speed $\tau > 0$ if there exists $n < q$, $C_q > 0$ and $\lambda > 0$ such that

1. $\lambda|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \lambda^{-1}|\xi|^2$ *for all* $\xi \in \mathbb{R}^n$,
2. $\|a^{ij} - \delta^{ij}\|_{1,q,-\tau} + \|b^i\|_{q,-1-\tau} + \|c\|_{\frac{q}{2},-\tau-2} \leq C_q$.

We remark that since $q > n$ then the a^{ij} are in fact Hölder and

$$|a^{ij} - \delta^{ij}| = o(r^{-\tau}).$$

Exercise 3.2 Show that an elliptic operator asymptotic to the Laplacian, viewed as $P : W^{2,p}_\delta \to L^p_{\delta-2}$ is bounded for all $p \leq q$.

Exercise 3.3 Show that Δ_g is asymptotic to the Laplacian if and only if $g - \delta \in W^{1,q}_{-\tau}$, for some $q > n$ and $\tau > 0$, and g is uniformly equivalent to δ.

We have an analogue of the elliptic regularity in this setting. That is to morally we gain two derivatives.

Theorem 3.2.2 *Let P an elliptic operator asymptotic to the Laplacian, $1 < p \leq q$ and $\delta \in \mathbb{R}$, there exists $C > 0$, depending on n, p, q, δ, C_q and λ, such that if $u \in L^p_\delta$ and $P(u) \in L^p_{\delta-2}$ then $u \in W^{2,p}_\delta$ and*

$$\|u\|_{2,p,\delta} \leq C(\|Pu\|_{p,\delta-2} + \|u\|_{p,\delta}).$$

Exercise 3.4 Prove the theorem using classical elliptic regularity and the scaling argument.

Corollary 3.2.3 *Under the hypothesis of the previous theorem. If moreover $P(u) = 0$ then $u \in W^{2,q}_\delta$.*

Our goal is to inverse such an operator, as we are able to do on a domain of a compact manifold for the Laplacian. But since the manifold is not compact and with our Dirichlet boundary condition, then it is possible that our operator get some Kernel.

In fact even the flat Laplacian get some kernel on $\mathbb{R}^n$, which are nothing else than the harmonic functions. Let us describe a bit more precisely this set.

The only harmonic function on $\mathbb{R}^n \setminus \{0\}$ which decrease as some r^γ are given by the harmonic polynomials and the derivatives of teh Green functions $\frac{1}{r^{n-2}}$. see[**?**]

Hence the order of decreasing are $E = \mathbb{Z} \setminus \{-n-3, \dots, -1\}$. For instance we can't inverse Δ on L^p_δ with $\delta \in E$. Indeed if $\delta = 0$, then $\Delta 1 = 0$, of course $1 \notin L^p_0$, but we can approximate it by $u_n \in C^\infty_c(B(0, n+1), [0,1])$ such that $u_n = 1$ on $B(0,n)$ and uniformly bounded laplacian. Then, if Δ where invertible, by application of the open map theorem, we will have

$$\|u\|_{p,0} \leq C\|\Delta u\|_{p,-2},$$

for some $C > 0$. But

$$\|u_n\|_{p,0} \sim \ln(n)^{\frac{1}{p}}$$

and

$$\|\Delta u_n\|_{p,-2} \leq \left(\int_n^{n+1} \sigma^{2p-n} r^{n-1}\, dr\right)^{\frac{1}{p}} \leq C \leq \left(\int_n^{n+1} r^{2p-1}\, dr\right)^{\frac{1}{p}} \leq C n^{2-\frac{1}{p}}.$$

But as soon as our decreasing exponent is far from one of the critical value E, then Δ is invertible.

Theorem 3.2.4 *If $\delta \notin E$ and $1 < p < +\infty$ then*

$$\Delta : W'^{2,p}_\delta \to L'^p_\delta$$

is an isomorphism and there exists $C > 0$ depending only on n, p and δ such that

$$\|u\|_{W'^{2,p}_\delta} \leq C\|\Delta u\|_{L'^p_{\delta-2}}.$$

This result is a direct consequence of the following theorem.

Theorem 3.2.5 *Let $a, b \in \mathbb{R}$ and $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ a kernel defined as follows*

$$K(x,y) = \frac{1}{|x|^a |x-y|^{n-2} |x|^b}.$$

Then the map $u \mapsto K * u$ is continuous from L^p onto itself.

Proof of theorem 3.2.4:proof in construction, do not read

Proof of theorem 3.2.5:

Claim 1: let $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that $k \geq 0$, homogeneous of degree $-\beta$ and such that for some $p \geq 1$

$$J = \int_0^\infty K(1,t) t^{\frac{n}{p}+n-\beta-1}\, dt < +\infty.$$

If

$$U(f)(x) = \int_{\mathbb{R}^n} k(|x|,|y|) f(y)\, dy$$

then

$$\|U(f)\|_p \leq C(J)\|f\|_p.$$

Let $x \in \mathbb{R}^n$ such that $|x| = R$, then

$$U(f)(x) = \int_{S^{n-1}} \int_0^\infty k(R,r) f(r,\xi)\, r^{n-1} dr d\xi = \int_{S^{n-1}} \int_0^\infty R^{n-\beta} k(1,t) f(Rt,\xi)\, t^{n-1} dt d\xi \tag{3.4}$$

we set $U(f)(R,\xi) = \int_0^\infty R^{n-\beta} k(1,t) f(Rt,\xi)\, t^{n-1} dt$. But

$$\left(\int_0^\infty |f(Rt,\xi)|^p\, t^{n-1} dt\right)^{\frac{1}{p}} = R^{\frac{-n}{p}} \left(\int_0^\infty |f(t,\xi)|^p\, t^{n-1} dt\right)^{\frac{1}{p}} \tag{3.5}$$

and

$$\left(\int_0^\infty |U(f)(R,\xi)|^p\, R^{n-1} dR\right)^{\frac{1}{p}} = \int_0^\infty |U(f)(R,\xi)| h(R)\, R^{n-1} dR \tag{3.6}$$

where $h(R) = \frac{|U(f)(R,\xi)|^{p-1}}{\|U(R,\xi)\|_p^{1-\frac{1}{p}}}$ with $\int_0^\infty h^{p'}\, R^{n-1} dR = 1$.

Then

$$\begin{aligned}
\left(\int_0^\infty |U(f)(R,\xi)|^p\, R^{n-1} dR\right)^{\frac{1}{p}} &= \int_0^\infty |U(f)(R,\xi)| h(R)\, R^{n-1} dR \\
&= \int_0^\infty \left| \int_0^\infty R^{n-\beta} k(1,t) f(Rt,\xi)\, t^{n-1} dt \right| h(R)\, R^{n-1} dR \\
&\leq \int_0^\infty k(1,t) t^{n-1} \int_0^\infty |f(Rt,\xi)| h(R) R^{n-\beta}\, R^{n-1} dR\, dt \\
&\leq \int_0^\infty k(1,t) t^{n-1} \left(\int_0^\infty |f(Rt,\xi)|^p R^{p(n-\beta)}\, R^{n-1} dR\right)^{\frac{1}{p}} t^{n-1} dt \\
&= \int_0^\infty k(1,t) t^{n-1-\frac{n}{p}} \left(\int_0^\infty |f(R,\xi)| R^{n-\beta}\, R^{n-1} dR\right)^{\frac{1}{p}} t^{n-1} dt \\
&= J \left(\int_0^\infty |f(Rt,\xi)| R^{n-\beta}\, R^{n-1} dR\right)^{\frac{1}{p}}
\end{aligned} \tag{3.7}$$

Finally

$$
\begin{aligned}
|U(f)(x)|^p &= \left(\int_{S^{n-1}} U(f)(R,\xi)\, d\xi\right)^p \\
&\leq \omega_{n-1}^{p-1} \int_{S^{n-1}} |U(f)(R,\xi)|^p\, d\xi
\end{aligned}
\tag{3.8}
$$

then

$$
\begin{aligned}
\|U(f)\|_p &\leq \omega_{n-1}^{p'} \left(\int_0^\infty \int_{S^{n-1}} |U(f)(R,\xi)|^p\, d\xi R^{n-1} dR\right)^{\frac{1}{p}} \\
&\omega_{n-1}^{p'} J^{\frac{1}{p}} \left(\int_0^\infty \int_{S^{n-1}} |f(R,\xi)|^p\, d\xi R^{n-1} dR\right)^{\frac{1}{p}}
\end{aligned}
\tag{3.9}
$$

which prove the desired result. □

Corollary 3.2.6 *Let $u \in L^p_\delta$ harmonic with $p > 1$ and $\delta \notin E$. We set $k^-(\delta) = \max\{k \in E \,|\, k < \delta\}$, if $k^-(\delta) < 0$ then $u \equiv 0$ else u is an harmonic polynomial of degree $k^-(\delta)$.*

Proof :
Since u is harmonic, by classical regularity, we know that $u \in C^\infty(\mathbb{R}^n)$. If $h_{k^-(\delta)} < 0$ we set $h_{k^-(\delta)} \equiv 0$ else let us consider the Taylor expansion at 0,

$$
u(x) = h_{k^-(\delta)}(x) + O(|x|^{k^-(\delta)+1}).
$$

Since $\Delta u(0) = 0$ we necessary have that $h_{k^-(\delta)}$ is an harmonic polynomial. In any case we easily check that $u - h_{k^-(\delta)} \in L'^p_\delta$. Hence thanks to theorem **??** we get that $u \equiv h_{k^-(\delta)}$ which proves the result. □

□

Theorem 3.2.7 *Let P asymptotic to the Laplacian with speed $\tau > 0$ and $q > n$. If $1 < p \leq q$ and $\delta \notin E$, then*

$$
P : W^{2,p}_\delta \to L^p_{\delta-2}
$$

has a finite dimensional kernel and closed range. Moreover there exists $C > 0$ and $R > 0$, which depends only on P, n, p and δ such that

$$
\|u\|_{2,p,\delta} \leq C \left(\|P(u)\|_{p,\delta-2} + \|u\|_{L^p(B_R)}\right). \tag{3.10}
$$

Proof of the theorem 3.2.7:
Let us set, for $X \subset \mathbb{R}^n$,

$$
\|P - \Delta\|_{X,2,p,\delta} \sup_{u \in W^{2,p}_\delta(X)\setminus\{0\}} \frac{\|(P-\Delta)(u)\|_{p,\delta-2}}{\|u\|_{2,p,\delta}}.
$$

Let $u \in W^{2,p}_\delta$ such that $supp(u) \subset \mathbb{R}^n \setminus B_R = E_R$, then, by Hölder inequality an Sobolev injection,

$$\begin{aligned}\|(P-\Delta)(u)\|_{p,\delta-2} &\leq \sup_{E_R} |a^{ij}-\delta^{ij}| \|\nabla^2 u\|_{p,\delta-2} + \|b_{|E_R}\|_{n,-1}\|\nabla u\|_{p^*,\delta-1} + \|c_{|E_R}\|_{\frac{n}{2},-2}\|u\|_{2,p^{**},\delta} \\ &\leq \left(\sup_{E_R} |a^{ij}-\delta^{ij} + \|b_{|E_R}\|_{q,-1-\tau} + \|c_{|E_R}\|_{\frac{q}{2},-2-\tau}\right) \|u\|_{2,p,\delta}.\end{aligned} \tag{3.11}$$

Then $\|P-\delta\|_{E_R,2,p,\delta} = o(1)$ as $R \to +\infty$. Here we implicitly assume that $p < n$ but the proof can be easily adapted to $n \geq p \geq q$.

Let $u \in W^{2,p}_\delta$ and $\chi \in C^\infty_c(B_2,[0,1])$ such that $\chi \equiv 1$ on B_1. We set $\chi_R(x) = \chi\left(\frac{x}{R}\right)$ and $u = u_1 + u_2 = \chi_R u + (1-\chi_R)u$ then, thanks to theorem 3.2.4, we get

$$\begin{aligned}\|u_2\|_{2,p,\delta} &\leq C\|\Delta u_2\|_{p,\delta-2} \\ &\leq C(\|(\Delta - P)(u_2)\|_{p,\delta-2} + \|P(u_2)\|_{p,\delta-2}) \\ &\leq C(\|(\Delta-P)\|_{E_R,p,\delta}\|u_2\|_{2,p,\delta} + \|P(u_2)\|_{p,\delta-2})\end{aligned} \tag{3.12}$$

moreover

$$\begin{aligned}\|P(u_2)\|_{p,\delta-2} &\leq \|P(u)\|_{p,\delta-2} + \|2a^{ij}\partial_i\partial_j\chi + (a^{ij}\partial_{ij}\chi + b^i\partial_i\chi)u\|_{p,\delta-2}) \\ &\leq \|P(u)\|_{p,\delta-2} + \|u_{|B_{2R}}\|_{1,p,\delta})\end{aligned} \tag{3.13}$$

Finally, taking R big enough,

$$\|u_2\|_{2,p,\delta} \leq C(\|P(u)\|_{p,\delta-2} + \|u_{B_{2R}}\|_{1,p,\delta}).$$

Hence, using some interpolation inequality, we get

$$\|u_2\|_{p,\delta} \leq C(\|P(u)\|_{p,\delta-2} + \|u_{|B_{2R}}\|_p)$$

$$\|u_1+u_2\|_{2,p,\delta} \leq C(\|P(u)\|_{p,\delta-2} + \|u_{|B_{2R}}\|_p)$$

which proves (3.10).
Hence let $u_k \in \ker(P)$ with $\|u_k\|_{2,p,\delta} = 1$. Using the compactness of the Sobolev embedding, we can extract a sequence which converges[1] into $L^p(B_R)$, then using (3.10), we get that this sequence is a Cauchy sequence into $W^{2,p}_\delta$ and then converges. Hence the unit ball of $\ker(P)$ is compact, which implies that $\ker(P)$ is finite dimensional by the Riesz theorem.
let us denote $Z \subset W^{2,p}_\delta$ such that $Z \oplus \ker(P) = W^{2,p}_\delta$, hence Z is closed. Moreover $P_{|Z}$ satisfies

$$\|u\|_{2,p,\delta} \leq C\|P_{|Z}(u)\|_{p,\delta-2}.$$

Indeed let u_k such that $\|u_k\|_{2,p,\delta} = 1$ and $\|P(u_k)\|_{p,\delta-2} \to 0$, as for the previous paragraph, we show that up to a subsequence u_k converges to $u \in Z$. Since P is continuous we have

[1] Here we use the fact that $\overline{B_R}$ is compact.

$P(u) = 0$ and $\|u|_{2,p,\delta} = 1$, which leads to a contradiction. Hence, let $f_k = P(u_k)$ a converging sequence, then by (3.10), u_k is bounded and then converges, up to extraction, to u such that $f = P(u)$, by continuity of P, then $Im(P)$ is closed. □

Let us remark, by Hölder inequality, that

$$(L^p_\delta)' = L^{p'}_{-n-\delta}$$

and then

$$(W^{k,p}_\delta)' = W^{-k,p'}_{-n-\delta}.$$

Now let assume that formal adjoint of P,

$$P^* : W^{0,p'}_{2-n-\delta} \to W^{-2,p'}_{-n-\delta}$$

is also asymptotic to the Laplacian. In this case, thanks to the previous theorem, P and P^* are Fredholm and we can use the Fredholm theory. Let us denote

$$N(P,\delta) = \dim\ker(P : W^{2,p}_\delta \to L^p_{\delta-2}).$$

We remark that thanks to theorem 3.2.7, this independent of $1 < p \leq q$. Thanks to theorem 3.2.7 $\ker(P^*) \subset W^{2,p'}_{2-n-\delta}$ hence this kernel is the same than for the map

$$P^* : W^{2,p'}_{2-n-\delta} \to L^{p'}_{-n-\delta}.$$

which gives

$$\dim co\ker(P) = \dim\ker(P^*) = N(P^*, 2-n-\delta)$$

. In that case we can define the index of the operator P by

$$i(P,\delta) = N(P,\delta) - N(P^*, 2-n-\delta).$$

Thanks to the index theory, see [1] chapter 3 or [17], we get :

$$i(P,\delta) = i(\Delta,\delta).$$

Let us now focus on $P = \Delta_g$ with $P^* = \Delta_g = P$, where the pairing is determined with respect to dv_g.

Proposition 3.2.8 *Let g a uniformly positive defined metric on $\mathbb{R}^n$ such that $g-\delta \in W^{1,q}_{-\tau}$ with $\tau > 0$ and $q > n$. If $\delta \notin E$ then*

$$N(\Delta_g,\delta) = N(\Delta,\delta).$$

Proof of proposition 3.2.8:
It suffices to show the proposition for $\delta < 0$. Indeed, thanks to the invariance of the index, we get

$$N(\Delta_g,\delta) - N(\Delta_g, 2-n-\delta) = N(\Delta,\delta) - N(\Delta, 2-n-\delta),$$

which will imply the result for $\delta \geq 0$.
If $\Delta_g u = 0$ then $u \in W^{2,p}_\delta$ for all p by regularity and then $u = o(1)$ by Sobolev injection. Finally using the maximum principle we have $u = 0$ and we conclude using proposition 3.2.4. □

3.3 Asymptotically flat manifolds

Definition 3.3.1 *A (M, g) complete Riemannian manifold, with $g \in W^{1,q}_{loc}(M)$ for some $q > n$ is said Asymptotically flat (with one end) if there exists $K \subset M$ compact and $\phi : M \setminus K \to \mathbb{R}^n \setminus B_1$ a diffeomorphism such that*

- *$\phi_*(g)$ is a uniformly positive defined metric, i.e. there exists $\lambda > 1$ such that*

$$\frac{1}{\lambda}|\xi|^2 \leq g_{ij}(x)\xi^i\xi^j \leq \lambda|\xi|^2 \ \forall x \in \mathbb{R}^n \setminus B_1 \ \forall \xi \in \mathbb{R}^n.$$

- $$\phi_*(g)_{ij} - \delta_{ij} \in W^{1,q}_{-\tau}(\mathbb{R}^n \setminus B_1)$$

  for some $\tau > 0$ called the decreasing rate.

Important remark: In this chart we defined σ and we remark that the definition of L^q_δ is independent of the chart by i). But $W^{k,q}_\delta$ depends on the chart ϕ, since the partial derivatives will depend on the choice of coordinates. It will be denoted $W^{k,q}_\delta(\phi)$. In the rest of the section, we will call such a chart **a structure at infinity**. Once the structure at infinity is chosen we naturally extend the definition of $W^{k,q}_\delta(\mathbb{R}^n \setminus B_1)$ to $W^{k,q}_\delta(M)$ since on the compact part all choices of chart defined the same structure.

Of course we can defined a multiple ends asymptotically flat structure on a manifold. But since analysis phenomena are determined by the behavior at infinity, it is very easy to isolate each ends and to consider that there is only one.

Thanks to the previous section we get the following result.

Theorem 3.3.2 *Let (M, g) an asymptotically flat manifold with a structure at infinity $\phi : M \setminus K \to \mathbb{R}^n \setminus B_1$ with decay rate τ. If $\delta \notin E$ then*

$$\Delta_g : W^{2,q}_\delta(\phi) \to L^q_{\delta-2}$$

is Fredholm. Moreover Δ is

$$\begin{cases} \text{if } \delta > 2-n \text{ then } \Delta_g \text{ is surjective,} \\ \text{if } 2-n < \delta < 0 \text{ then } \Delta_g \text{ is bijective,} \\ \text{if } \delta < 0 \text{ then } \Delta_g \text{ is injective.} \end{cases} \tag{3.14}$$

The goal of the next theorem is to show that two different asymptotically flat structures differ from an isometry up to a small error.

Theorem 3.3.3 *Let (M, g) an asymptotically flat manifold with a structure at infinity $\phi : M \setminus K \to \mathbb{R}^n \setminus B_1$ with decay rate $\tau > 0$. Let $1 < \eta < 2$ then there exists $y^i \in L^q_\eta$ such that*

$$\Delta_g y^i = 0$$

and

$$(x^i - y^i) \in W^{2,q}_{1-\tau'}(\phi),$$

where $\tau' = min(\tau, n-1-\varepsilon)$, for any $\varepsilon > 0$ which implies

$$\begin{cases} |x^i - y^i| = o(r^{1-\tau'}), \\ g(\partial_{x^i}, \partial_{x^j}) - g(\partial_{y^i}, \partial_{y^j}) = o(r^{-\tau'}). \end{cases} \tag{3.15}$$

Moreover $\{1, y^1, \dots, y^n\}$ is basis of $\mathbb{H}_1 = \{u \in L^q_\eta \,|\Delta_g u = 0\}$[2].

Proof of theorem 3.3.3:
First of all, we extend the coordinates x^i to M to some smooth function. Then we remark that $\Delta_g x^i \in L^q_{-1-\tau}$, indeed

$$\Delta_g x^i = \frac{\partial_j(g^{ij}\sqrt{|g|})}{\sqrt{|g|}} \in L^q_{-1-\tau}(\mathbb{R}^n \setminus B_1).$$

Moreover, thanks to theorem 3.3.4, we know that

$$\Delta_g : W^{2,q}_{1-\tau'} \to L^q_{-1-\tau'}$$

is surjective since $\tau' < n-1$. Hence there exists $v_i \in W^{2,q}_{1-\tau'}$ such that

$$\Delta_g v_i = \Delta x^i.$$

Hence $y^i = x^i - v^i$ is harmonic with the appropriate decreasing. Moreover $x^i \in L^q_\eta$, since

$$\int_{\mathbb{R}^n \setminus B_1} |x^i|^q r^{-q\eta - n} dx \leq \int_{\mathbb{R}^n \setminus B_1} r^{q(1-\eta)-n} dx.$$

And, since $1 - \tau' < 1$, $v^i \in L^q_\eta$, hence $y^i \in L^q_\eta$ and then $y^i \in \mathbb{H}_1$. Thanks to corollary 3.2.6 we have $\dim(H^1) = n+1$ and we trivially check that 1 and the y^i are free, it achieves the proof of the theorem. □

Corollary 3.3.4 *Let (M, g) an asymptotically flat manifold with two structures at infinity $\phi, \psi : M \setminus K \to \mathbb{R}^n \setminus B_1$ with decay rate τ_ϕ and τ_ψ. There exists $O, a \in O(n) \times \mathbb{R}^n$ such that*

$$x^i - (O^i_j z^j + a^i) \in W^{2,q}_{1-\tau}$$

which implies

$$|x^i - (O^i_j z^j + a^i)| = o(r^{1-\tau}),$$

where $\tau = \min(\tau_\phi, \tau_\psi, n-1-\varepsilon)$ for any $\varepsilon > 0$, $x = \phi^{-1}$ and $z = \psi^{-1}$.

[2] This space (as L^q_η) doesn't depend the structure

Proof:

So let denote y and w the harmonic coordinates associated to ϕ and ψ. Thanks to the previous theorem $\{1, y^1, \dots, y^n\}$ and $\{1, w^1, \dots, w^n\}$ are two basis of $\mathbb{H}^1$, hence there exists $O, a \in GL(n) \times \mathbb{R}^n$ such that

$$y^i = O^i_j w^j + a^i.$$

But since $\frac{\partial}{\partial y^i}$ and $\frac{\partial}{\partial w^i}$ are asymptotically two orthonormal basis of the flat metric, since $|g - \delta| = o(1)$, hence we get that $O \in O(n)$.

Finally writing,

$$x^i - (O^i_j z^j + a^i) = (x^i - y^i) + (O^i_j (z^j - w^j),$$

we get the desired estimate. □

This estimate is the best possible and it is determined by the Ricci curvature as shown by the next theorem, see also [6].

Corollary 3.3.5 *Let (M, g) an asymptotically flat manifold with a structure at infinity $\phi : M \setminus K \to \mathbb{R}^n \setminus B_1$ with decay rate τ such that $(\phi_* g - \delta) \in W^{2,q}_{-\tau}(\mathbb{R}^n \setminus B_1)$ for $q > n$ and such that*

$$Ric_g \in L^q_{-2-\eta}(M) \text{ for some } \eta > \tau \text{ and } \eta \notin E.$$

Then there exists a structure at infinity $\Theta : M \setminus K' \to \mathbb{R}^n \setminus B_1$ such that $(\Theta_* g - \delta) \in W^{2,q}_{-\eta}(\mathbb{R}^n \setminus B_1)$.

Proof :
In harmonic coordinates, we get

$$(Ric_g)_{ij} = -\frac{1}{2} g^{kl} \partial_{kl} g_{ij} + Q(g, \partial g)$$

where $Q(g, \partial g)$ is a quadratic expression in ∂g. Hence we easily check that thanks to our hypothesis $g^{kl} \partial_{kl} g_{ij} \in L^q_{-\eta-2}$ then bootstraping the regularity result we get that the asymptotic structure Θ given by the harmonic coordinates satisfies the desired estimate. □

3.4 Mass of an asymptotically flat manifold

In exercise 1.12 we have proved using only the divergence free structure of the stress energy tensor that the total energy

$$\int_{x_0 = t} T_{00} \, dx$$

is conserved through time. Hence if we get a solution of $(M, \tilde{g})$ of the Einstein equation, such that $(M, \tilde{g})$ split as $(\mathbb{R} \times N, -V^2 dt^2 + g)$, then the quantity

$$\int_N T_{00}\, dv_g$$

is a good candidate to define the total mass of the universe. Then using the first constraint equation in the symmetric case, we can write that

$$T_{00} = \frac{R_g}{16\pi}.$$

Let us assume moreover that (N, g) is asymptotically flat, then as for the Newtonian gravity, we would like to express this quantity as a limit at infinity. Moreover at infinity all slices will be flat and then the assumption to be symmetric will become trivial and the definition of the total mass will depends only on (N, g). But Thanks to (2.15) of [9], we have the following formula for the scalar curvature

$$\begin{aligned} R_g &= ({g_i}^{j,i} - {g_i}^{i,j})_{,j} + \Gamma * \Gamma \\ &= div(X) + \Gamma * \Gamma \end{aligned} \tag{3.16}$$

where X is the vector field given by ${g_i}^{j,i} - {g_i}^{i,j}$. Hence, if we get a vector field Y such that

$$\int_{M \setminus B_R} R_g\, dv_g = \int_{\partial B_R} g(Y, \nu)\, d\sigma + o(1),$$

for all R big enough, then it must satisfy for all $R' > R$ big enough

$$\int_{B_{R'} \setminus B_R} R_g\, dv_g = \int_{\partial B_{R'}} g(Y, \nu)\, d\sigma_g - \int_{\partial B_R} g(Y, \nu)\, d\sigma_g + o(1). \tag{3.17}$$

Which is exactly what is done by X if we assume that the asymptotic structure is such that $(\phi_*(g) - \delta) \in W^{2,q}_{-\tau}$ for $q > n$ and $\tau > \frac{n-2}{2}$. Since

$$|\Gamma * \Gamma| = o(r^{-2(1+\tau)})$$

with $2(1 + \tau) > n$. All this heuristic leads to the following theorem

Theorem 3.4.1 (Bartnik 1986) *Let (N, g) be a asymptotically flat manifold with $R_g \in L^1$ and and an structure at infinity $\phi : M \setminus K \to \mathbb{R}^n \setminus B_1$ such that $(\phi_*(g) - \delta) \in W^{2,q}_{-\tau}$ for $q > n$ and $\tau > \frac{n-2}{2}$. Then the following limit exists*

$$m(g) = \lim_{R \to +\infty} \frac{1}{16\pi} \int_{\partial B_R} ({g_{ij}}^{,i} - {g_i}^{i}{}_{,j}) \nu^j\, d\sigma,$$

and is independent of the structure at infinity.

Proof of theorem 3.4.1:
The fact that the limit is well defined is a direct consequence of (3.17) and the hypothesis of the theorem.

The rest of the proof is done assuming that $\tau < n-1$, else we have to consider $\tau' = n - \frac{1}{2}$ which gives enough decreasing.

Consider now two systems of coordinates x^i and y^i given by two different structures at infinity ϕ^1 and ϕ^2, thanks to theorem corollary 3.3.3, up to composition with an isometry, we have

$$x^i - y^i \in W^{3,q}_{1-\tau}$$

since $g \in W^{2,q}_{-\tau}$.[3] Here all function are seen in the chart ϕ^1. More precisely let denote $g^i = \phi^i_*(g)$ in $\mathbb{R}^n \setminus B_1$ and $\phi = \phi^1 \circ (\phi^2)^{-1}$, as in figure 3.1.

$$\begin{tikzcd}
 & (\mathbb{R}^n \setminus B_1, y, g_2) \arrow[d, "\phi"] \\
M \setminus K \arrow[ur, "\phi_2"] \arrow[r, "\phi_1"'] & (\mathbb{R}^n \setminus B_1, x, g_1)
\end{tikzcd}$$

Figure 3.1: Two structures at infinity

The previous estimate, written in coordinates, reads

$$v(x) = x - \phi(x) = o_2(r^{1-\tau}),$$

where $u = o_k(\sigma^\alpha)$ means $\partial_i u = o(r^{\alpha - i})$ for all $i \leq k$.

We want to compare

$$X^1 = \frac{\partial g^1_{ij}}{\partial x^i} - \frac{\partial g^1_{ii}}{\partial x^j}$$

and

$$X^2 \circ \phi \text{ where } X^2 = \frac{\partial g^2_{ij}}{\partial y^i} - \frac{\partial g^2_{ii}}{\partial y^j}.$$

But

$$g^1_{ij} - g^2_{ij} \circ \phi = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) - g\left(\phi_*\left(\frac{\partial}{\partial y^i}\right), \phi_*\left(\frac{\partial}{\partial y^j}\right)\right),$$

moreover

$$\phi_*\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial \phi}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{\partial v}{\partial x^i}.$$

[3] In fact we need an improve version of theorem 3.3.3 which assert that if $g \in W^{k,q}_{1-\tau}$ then $x^i - y^i \in W^{k+1,q}_{1-\tau}$, whose proof is left to the reader.

Finally

$$g^1_{ij} - g^2_{ij} \circ \phi = \frac{\partial v^j}{\partial x^i} + \frac{\partial v^i}{\partial x^j} + O(|\nabla v|^2).$$

Taking a derivative, we get

$$\begin{aligned}
\frac{\partial g^1_{ij}}{\partial x^l} - \frac{\partial g^2_{ij}}{\partial y^l} \circ \phi &= \frac{\partial}{\partial x^l}\left(g^1_{ij} - g^2_{ij} \circ \phi\right) + \left(\frac{\partial g^2_{ij}}{\partial y^k} \circ \phi \frac{\partial \phi^k}{\partial x^l} - \frac{\partial g^2_{ij}}{\partial y^l} \circ \phi\right) \\
&= \frac{\partial^2 v^j}{\partial x^l \partial x^i} + \frac{\partial^2 v^i}{\partial x^l \partial x^j} + O(|\nabla v| \nabla g|) \\
&= \frac{\partial^2 v^j}{\partial x^l \partial x^i} + \frac{\partial^2 v^i}{\partial x^l \partial x^j} + O(r^{-1-2\tau}).
\end{aligned} \tag{3.18}$$

Finally

$$X^1 - X^2 \circ \phi = \left(\frac{\partial^2 v^j}{\partial^2 x^i} - \frac{\partial^2 v^i}{\partial x^j \partial x^i}\right) + O(r^{-1-2\tau})$$

Remarking that

$$div\left(\frac{\partial^2 v^j}{\partial^2 x^i} - \frac{\partial^2 v^i}{\partial x^j \partial x^i}\right) = 0$$

we get

$$\int_{\partial B_R} \langle X^1, \nu \rangle \, d\sigma = \int_{\partial B_R} \langle X^2 \circ \phi, \nu \rangle \, d\sigma + o(1),$$

which proves the desired property. □

Let us remark that *a priori we need* $\nabla g = O(1/r^2)$ for the integral to be convergent, but the theorem requires much less due to some cancellation phenomena exhibit by in proof.

Exercise 3.5

1. Compute the mass of the Schwarzschild metric

$$g = \left(1 + \frac{m}{2r}\right)^4 \delta.$$

2. More generally, show that if

$$g = \left(1 + \frac{m}{2r}\right) \delta + o_2\left(\frac{1}{r^2}\right),$$

   then m(g)=m.

3. (Harder) Prove that if the asymptotically flat manifold is given by $(M \backslash \{x\}, G(x, \,.\,)^{\frac{4}{n-2}} g)$ where (M, g) is a compact manifold and G the Green function of the conformal

Laplacian, see end of chapter 2, then $m(g) = \beta(x, x)$ where β is defined as the regular part of G, i.e.

$$G(x, y) = \frac{1}{(n-2)\omega_{n-1} d_g(x, y)^{n-1)}} + \beta(x, y).$$

Chapter 4

The Plateau problem

The goal of this chapter is to solve the following problem:
Let (N, g) be a complete Riemannian 3-manifold and $\Gamma \subset N$ a smooth Jordan curve, can we find a disc Σ such that

$$\begin{cases} \partial\Sigma = \Gamma \\ \forall \Sigma' \text{ such that } \partial\Sigma' = \Gamma \text{ then } |\Sigma| \leq |\Sigma'| \end{cases}$$

This problem has been raised by Joseph-Louis Lagrange in 1760. However, it is named after Joseph Plateau (end of 19th century) who experimented it with soap films. Here we mainly focus on $N = \mathbb{R}^3$ where we are going to met the main difficulty from the point of view of calculus of variations in a second section we will explain how to generalize it to a general Remannian manifold, especially how to deal with the regularity issue in this setting.

4.1 Equation and existence

Let $u : \mathbb{D} \to \mathbb{R}^3$ a C^2-immersion and $\Sigma = u(\mathbb{D})$, then we naturally set

$$|\Sigma| = \int_{\mathbb{D}} |u_x \wedge u_y| \, dz.$$

Let $\phi \in C_c^\infty(\mathbb{D})$ and $\Sigma_t = u^t(\mathbb{D})$ where $u^t = u + t\phi\vec{n}$ with $\vec{n} = \frac{u_x \wedge u_y}{|u_x \wedge u_y|}$ is the Gauss map of the surface.

Then

$$\begin{aligned} u_x^t &= u_x + t\phi\vec{n}_x + t\phi_x\vec{n} \\ u_y^t &= u_y + t\phi\vec{n}_y + t\phi_y\vec{n} \end{aligned} \tag{4.1}$$

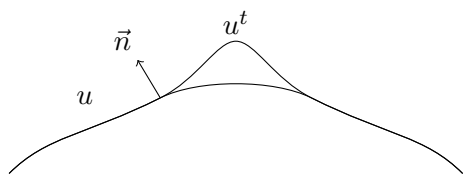

Figure 4.1: Small perturbation of a surface

Then

$$\begin{aligned}
|u_x^t \wedge u_y^t| &= \left(\det\begin{pmatrix}\langle u_x^t, u_x^t\rangle & \langle u_x^t, u_y^t\rangle \\ \langle u_y^t, u_x^t\rangle & \langle u_y^t, u_y^t\rangle\end{pmatrix}\right)^{\frac{1}{2}} \\
&= (\det(I - t\phi \mathrm{II}))^{\frac{1}{2}} + o(t) \\
&= \left(\det(I(Id - tI^{-1}\mathrm{II}))\right)^{\frac{1}{2}} + o(t) \\
&= |u_x \wedge u_y| \left(1 - \frac{t\phi}{2} tr(I^{-1}\mathrm{II}) + o(t)\right)
\end{aligned} \tag{4.2}$$

where

$$I = \begin{pmatrix}\langle u_x, u_x\rangle & \langle u_x, u_y\rangle \\ \langle u_y, u_x\rangle & \langle u_y, u_y\rangle\end{pmatrix}$$

and

$$II = -\begin{pmatrix}\langle u_x, \vec{n}_x\rangle & \langle u_x, \vec{n}_y\rangle \\ \langle u_y, \vec{n}_x\rangle & \langle u_y, \vec{n}_y\rangle\end{pmatrix},$$

where I and II are respectively the first and second fundamental form of the surface. Finally

$$|\Sigma^t| = \int_{\mathbb{D}} (1 - t\phi H)|u_x \wedge u_y|\, dz + o(t),$$

where H is the mean curvature of the surface. Hence the solution of our problem must satisfies

$$H \equiv 0,$$

know as minimal surface equation. This equation becomes much simpler in a good choice of coordinates, namely the conformal coordinates whose existence is insured by the following theorem.

Theorem 4.1.1 (Gauss-Chern, see chap 9 of [28] and [4]) *Let $u : \mathbb{D} \to \mathbb{R}^3$ a C^2-immersion, there exists φ a C^2 diffeomorphism of $\mathbb{D}$ such that $\tilde{u} = u \circ \varphi$ be conformal, i.e.*

$$\begin{cases} |\tilde{u}_x|^2 = |\tilde{u}_y|^2, \\ \langle \tilde{u}_x, \tilde{u}_y\rangle = 0. \end{cases} \tag{4.3}$$

The optimal hypothesis for the existence of conformal coordinates is that $\nabla\vec{n} \in L^2$, see 5.4 of [15].

Hence in conformal coordinates, the minimal surface equation reduces to

$$\langle u_x, \vec{n}_x\rangle + \langle u_y, \vec{n}_y\rangle = 0$$

That is to say

$$\langle \Delta u, \vec{n}\rangle = 0.$$

But, using (4.3)

$$\begin{aligned} \langle \Delta u, u_x\rangle &= \left(\frac{|u_x|^2}{2}\right)_x - \left(\frac{|u_y|^2}{2}\right)_x \\ &= 0 \\ &= -\langle \Delta u, u_y\rangle. \end{aligned} \tag{4.4}$$

Finally the equation is equivalent to

$$\begin{cases} \Delta u = 0, \\ u \text{ is conformal.} \end{cases} \tag{4.5}$$

In order to prove the existence of such a map, we naturally try to minimize the area functional

$$\mathcal{A}(u) = \int_{\mathbb{D}} |u_x \wedge u_y|\, dz$$

under the constraint $u(S^1) = \Gamma$. The main problem of this approach is that, since we deal with purely geometric quantities, the functional is invariant under reparametrization, i.e.

$$\mathcal{A}(u \circ \varphi) = \mathcal{A}(u),$$

for all diffeomorphisms φ of $\mathbb{D}$ which preserve the boundary. Hence there is no chance that an arbitrary minimizing sequence will converge. The crucial idea, due to Douglas and Radó, is to replace the area functional by a functional with a smaller invariance group. Since we know that in good coordinates the equation reduces to the harmonic equation, let us consider the Dirichlet functional

$$\mathcal{D}(u) = \frac{1}{2}\int_{\mathbb{D}} |\nabla u|^2\, dz,$$

but not with Dirichlet boundary data, rather some partially free boundary data, namely $u_{|S^1}$ is a monotone parametrization of Γ.

Exercise 4.1 Check the group of invariance of $\mathcal{D}$ is the Mobius group, i.e.

$$\mathcal{M} = \left\{ z \mapsto e^{i\theta}\frac{z+a}{1+\bar{a}z} \mid a \in \mathbb{D}, \theta \in \mathbb{R} \right\}.$$

Before going further in the analysis, we have to check that if u is a critical point of $\mathcal{D}$, it is also a critical point of $\mathcal{A}$. First we remark that, for any u we have

$$\mathcal{A}(u) \leq \mathcal{D}(u).$$

Theorem 4.1.2 *Let u be a critical point of $\mathcal{D}$ then u is conformal.*

Corollary 4.1.3 *Let u be a critical point of $\mathcal{D}$ then u is also a critical point of $\mathcal{A}$.*

Proof of corollary :
Thanks to the previous theorem, we know that u is conformal, then $\mathcal{D}(u) = \mathcal{A}(u)$. Since $\mathcal{D}'(u) = 0$ we necessarily get $\mathcal{A}(u) = 0$. □

Proof of theorem 4.1.2:
Let $\tau \in C^\infty(B(0,2), \mathbb{R}^2)$ such that $\langle \tau(x), x\rangle = 0$ for all $x \in S^1$. Then $\varphi_\varepsilon = Id + \varepsilon\tau$ defined a diffeomorphism of $\mathbb{D}$ onto its image for ε small enough. We set $u^\varepsilon = u \circ \varphi_\varepsilon$. We check that

$$u_\varepsilon = u + \varepsilon\langle u, \tau\rangle + o(\varepsilon)$$

and that u^ε satisfy the boundary condition at the first order. Moreover

$$\begin{aligned}
\mathcal{D}(u^\varepsilon) &= \frac{1}{2}\int_{\mathbb{D}} |\nabla u^\varepsilon|^2\, dz \\
&= \frac{1}{2}\int_{\mathbb{D}} |((\nabla u)\circ\varphi_\varepsilon).\nabla\varphi_\varepsilon|^2 \;\; dz \\
&= \frac{1}{2}\int_{\mathbb{D}} |((\nabla u)).(\nabla\varphi_\varepsilon\circ\varphi_\varepsilon^{-1})|^2 |\det(\nabla\varphi_\varepsilon^{-1}|)\, dz + o(\varepsilon),
\end{aligned} \tag{4.6}$$

the last inequality is a consequence of the fact $\mathbb{D}$ is preserve by φ_ε at first order. Moreover

$$\nabla\varphi_\varepsilon \circ \varphi_\varepsilon^{-1} = Id + \varepsilon\nabla\tau + o(\varepsilon)$$

and

$$\det(\nabla\varphi_\varepsilon^{-1}) = 1 - \varepsilon div(\tau) + o(\varepsilon).$$

Then

$$\begin{aligned}
\mathcal{D}(u^\varepsilon) &= \frac{1}{2}\int_{\mathbb{D}} |\nabla u|^2\, dz \\
&\quad + \varepsilon\int_{\mathbb{D}} (|u_x|^2\tau_x^1 + |u_y|^2\tau_y^2 + \langle u_x, u_y\rangle(\tau_x^2 + \tau_y^1) - \frac{1}{2}|\nabla u|^2 \mathrm{div}(\tau))\, dz + o(\varepsilon) \\
&= \frac{1}{2}\int_{\mathbb{D}} |\nabla u|^2\, dz + \varepsilon\int_{\mathbb{D}} \frac{1}{2}(|u_x|^2 - |u_y|^2)(\tau_x^1 - \tau_y^2) + \langle u_x, u_y\rangle(\tau_x^2 + \tau_y^1)\, dz \\
&+ o(\varepsilon) \\
&= \frac{1}{2}\int_{\mathbb{D}} |\nabla u|^2\, dz + \frac{\varepsilon}{2}\int_{\mathbb{D}} \Re(\phi\partial_{\bar z}\tau)\, dz + o(\varepsilon),
\end{aligned} \tag{4.7}$$

where

$$\phi(z) = (|u_x|^2 - |u_y|^2) + 2i\langle u_x, u_y\rangle,$$

is the Hopf differential of u. Since u is a critical point of $\mathcal{D}$, u is harmonic and then ϕ is holomorphic. Using again the fact that u is a critical point of $\mathcal{D}$, (4.7) gives, after integration by parts,

$$\int_{\partial\mathbb{D}} \Re(\phi\tau z)\, d\theta = 0.$$

Testing again $\tau = \alpha i z$ for any real function α on $\partial\mathbb{D}$, we get that

$$(z^2\phi)_{|S^1} \in \mathbb{R}.$$

Since $z^2\phi$ is holomorphic, by Liouville theorem, we get $z^2\phi$ is constant, and since $(z^2\phi)(0) = 0$ this gives $\phi \equiv 0$, which achieved the proof of the theorem. □

Theorem 4.1.4 *Let $u \in C^2(\mathbb{D}, \mathbb{R}^3)$ then for all $\varepsilon > 0$ there exists $\varphi : \mathbb{D} \to \mathbb{D}$ diffeomorphism such that $\tilde{u} = u \circ \varphi$ satisfies*

$$\mathcal{D}(\tilde{u}) \leq (1 + \varepsilon)\mathcal{A}(\tilde{u}).$$

Proof of theorem 4.1.4:
see appendix A of [30]. □

Corollary 4.1.5 *u is a minimizer of $\mathcal{A}$ if and only if it is minimizer for $\mathcal{D}$.*

In order to minimize $\mathcal{D}$, we have to kill the action of the Mobius group, which can be done fixing three points, thanks to the following exercice.

Exercise 4.2 Prove that for any triple $(p_1 = e^{i\theta_1}, p_2 = e^{i\theta_2}, p_3 = e^{i\theta_3})$ of S^1, with $\theta_1 < \theta_2 < \theta_3$, there exists a unique Mobius transformation Θ, such that $\Theta(1) = p_1$, $\Theta(i) = p_2$ and $\Theta(-1) = p_3$.

Now we consider $\gamma : S^1 \to \Gamma$ a parametrization of Γ by the arc-length, i.e. $|\dot{\gamma}| = 1$, and we set $p_j = \gamma(e^{\frac{ij\pi}{2}})$, for $j = 0, 1$ and 2. We define the following spaces,

$$C(\Gamma) = \{u \in H^1(\mathbb{D}, \mathbb{R}^3) \,|\, u_{|S^1} \text{ is a continuous monotone parametrization of } \Gamma\}.$$

and

$$C^*(\Gamma) = \{u \in C(\Gamma) \,|\, u(e^{\frac{ij\pi}{2}}) = p_j for j \in \{0, 1, 2\}\}.$$

Lemma 4.1.6 (Courant-Hilbert) *Let $u \in H^1(\mathbb{D}, \mathbb{R}^3)$, $z_0 \in S^1$ and $\delta \in (0, 1)$ then there exists $\rho \in [\delta, \sqrt{\delta}]$ such that, if s is the arc length of the arc $C_\rho = \partial B_\rho(z_0) \cap \mathbb{D}$, we get*

$$\int_{C_\rho} |u_s|^2\, ds \leq \frac{8\mathcal{D}(u)}{\rho|\ln(\rho)|}.$$

Proof of the lemma:
Thanks to Fubini theorem we get

$$\begin{aligned}
2\mathcal{D}(u) &\geq \int_{\mathbb{D}\cap B_{\sqrt{\delta}}(z_0)\setminus B_\delta(z_0)} |\nabla u|^2\, dz \geq \int_\delta^{\sqrt{\delta}} \int_{C_\rho} |u_s|^2\, ds d\rho \\
&\geq \inf_{\delta\leq\rho\leq\sqrt{\delta}} \left(\rho \int_{C_\rho} |u_s|^2)\, ds\right) \int_{C_\rho} \frac{d\rho}{\rho} \\
&\geq \inf_{\delta\leq\rho\leq\sqrt{\delta}} \left(\rho \int_{C_\rho} |u_s|^2)\, ds\right) \frac{|\ln(\delta)|}{2} \\
&\geq \inf_{\delta\leq\rho\leq\sqrt{\delta}} \left(\rho \int_{C_\rho} |u_s|^2)\, ds\right) \frac{|\ln(\rho)|}{2},
\end{aligned} \tag{4.8}$$

which proves the result. □

Theorem 4.1.7 *The injection $C^*(\Gamma) \subset C^0(S^1, \mathbb{R}^3)$ is compact.*

Proof of the theorem:
Let $u \in C^*(\Gamma)$, $\varepsilon > 0$ and $z_0 \in S^1$, we are going to show that there exists $\delta > 0$ depending only on ε, $\mathcal{D}(u)$ and the p_j such that, for any $z \in S^1$,

$$|z - z_0| < \delta \Rightarrow |u(z) - u(z_0)| < \varepsilon.$$

Then we derive the compactness by a direct application of the Arzela-Ascoli theorem.

Let $\delta_0 > 0$ such that any ball of $\mathbb{R}^3$ with radius smaller that δ_0 contains at most one p_j. We can also assume that

$$\varepsilon < \min_{i\neq j} \frac{d(p_i, p_j)}{4}.$$

Exercise 4.3 There exists $0 < \varepsilon_1 < \varepsilon$ such that for every $X, Y \in \Gamma$ such that $|X - Y| < \varepsilon_1$, there exists a unique $\tilde{\Gamma} \subset \Gamma$ joining x and y and included in a ball of radius $\frac{\varepsilon}{2}$.

Finally, let $0 < \delta < \delta_0$ such that

$$\frac{16\pi\mathcal{D}(u)}{\varepsilon_1^2} \leq |\ln(\delta)|.$$

Thanks to the Courant-Hilbert Lemma, there exists $\rho \in [\delta, \sqrt{\delta}]$ such that, if $\{x', y'\} = S^1 \cap \partial B_\rho(z_0)$, then

$$|u(x') - u(y')|^2 \leq \frac{8\pi\rho\mathcal{D}(u)}{\rho|\ln(\rho)|} \leq \varepsilon_1^2.$$

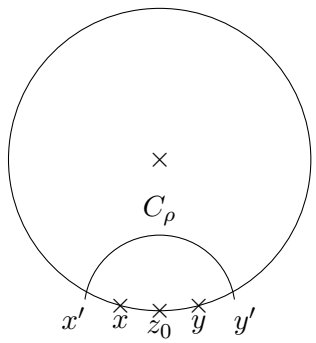

Figure 4.2: The preimage of $\tilde{\Gamma}$

Hence, thanks to the exercise 4.3, there exists an arc $\tilde{\Gamma} \subset B_{\frac{\varepsilon}{2}}$ joining $u(x')$ and $u(y')$.

It is clear from figure 4.2, that for any $x, y \in S^1 \cap B(z_0, \delta)$ then $u(x), u(y) \in B_{\frac{\varepsilon}{2}}$, which achieves the proof. □

Corollary 4.1.8 *$C^*(\Gamma)$ is closed for the weak-topology.*

Theorem 4.1.9 (Douglas-Rado 1936) *Let Γ a C^2 curve of $\mathbb{R}^3$ then there exists $u \in C(\Gamma)$ such that*

$$\begin{cases} \Delta u = 0 \\ u \text{ is conformal} \end{cases}$$

moreover u is a minimizer of $\mathcal{A}$ over $C(\Gamma)$.

Proof of theorem 4.1.9: Thanks to theorem 4.1.4, we know that

$$\inf_{u \in C(\Gamma)} \mathcal{A}(u) = \inf_{u \in C(\Gamma)} \mathcal{D}(u),$$

moreover, by conformal invariance of $\mathcal{D}$ and thanks to exercise 4.1, we have

$$\inf_{u \in C(\Gamma)} \mathcal{D}(u) = \inf_{u \in C^*(\Gamma)} \mathcal{D}(u).$$

Then we consider a minimizing sequence for $\mathcal{D}$ into $C^*(\Gamma)$. Since this last space is closed for the weak-topology, we extract a sequence which converge to $u \in C^*(\Gamma)$ which solves the problem. □

4.2 Immersion, Regularity and Generalization

The solution of theorem 4.1.9 is not yet a minimal surface, since we don't know if it is an immersion. In general a solution of the minimal surface equation can have a branched

point (a point where $rank(du) \neq 2$). For instance, $z \mapsto z^2$. But of course this is not a solution of the Plateau problem for the circle. In fact it suffices that the curve admit a projection onto a planar convex curved to exclude branch point and also get uniqueness at the same time, as proved by the following theorem.

Theorem 4.2.1 (theorem 1, section 4.9 [7]) *If Γ is a C^2 Jordan curve of $\mathbb{R}^3$ which possesses a one-to-one parallel projection onto a planar convex Jordan curve, then Γ bounds at most one minimal surface and it has no branch points.*

Here we even have uniqueness. Here is a similar version without uniqueness as exercise.

Exercise 4.4 [not very easy] Let Γ a smooth curve which satisfies

$$\int_\Gamma |k| dl < 4\pi$$

Then using the Gauss Bonnet formula prove that a minimal disc which bounds Γ has no interior branch point.

If we had the minimizing property, then with some comparaison argument we can also exclude branch point.

Theorem 4.2.2 (Theorem 5.8 of [30]) *Suppose that $u \in C(\Gamma)$ minimizes $\mathcal{D}$ in $C(\Gamma)$. Then u have no interior branch point. If in addition Γ is analytic then u has no boundary branch points, either.*

Then come the question of regularity, of course in the interior we are smooth since the parametrization is harmonic in conformal coordinates. At the boundary, the surface inherits of the regularity of the curves. Indeed the idea is to use a local chart to send the curve onto a line and then use the Schwarz reflexion principle to extend locally the map to a solution of an elliptic PDE. This way we prove that the maximal regularity is the one of the map which is given by the curve.

Theorem 4.2.3 (Theorem 5.1 of [30]) *If Γ is a $C^{k,\alpha}$ Jordan curve of $\mathbb{R}^3$ with $k \geq 1$ and $0 < \alpha < 1$ then any solution $u \in C(\Gamma)$ of*

$$\begin{cases} \Delta u = 0, \\ u \text{ is conformal}, \end{cases} \tag{4.9}$$

is in $C^{k,\alpha}(\overline{\mathbb{D}})$.

What is changed if we replace $\mathbb{R}^3$ by a smooth complete Riemannian manifold? For the existence part, the strategy is the same, we just have to be sure that $C(\Gamma) \neq \emptyset$. The fact that there is no branch point for the minimizer is conserved since it is a local

property. The main difference is in the regularity, in the interior. for the minimizer we can use some motonicity formula to get some Morrey estimate, see [20]. But, in a general setting, we have to speak about the regularity of the harmonic equation into a Riemannian manifold.In fact this equation becomes nonlinear, namely

$$\Delta u^k - \Gamma^k_{ij}(u)\nabla u^i \nabla u^j = 0.$$

However, the solution of this equation are smooth as soon as the manifold is. Which is very far from being easy, since the non-linearity is *a priori* in L^1. The proof relies on the existence of a good frame in which the equation is subject to some compactness by compensation phenomena. This subject is not developped here because it deserves its own book and also because this book has already been written by Helein [15].

4.3 Stability of minimal surfaces

In this section we are going to derive the first and the second variation for the area functional. Computation are made into exponential chart.

Let (N, g) a Riemannian manifold and $u^t : \mathbb{D} \to (N, g)$ a sequence of smooth immersions. We assume that

$$X = \left.\frac{\partial u^t}{\partial t}\right|_{t=0}$$

has compact support. Then

$$\mathcal{A}(u^t) = \int_{\mathbb{D}} \sqrt{\det({}^t du^t, du^t)}\, dz$$

with

$$\begin{aligned}
\frac{d\sqrt{\det({}^t du^t, du^t)}}{dt} &= \frac{1}{2} tr \frac{d({}^t du^t, du^t)}{dt} \sqrt{\det({}^t du^0, du^0)} \\
&= \frac{1}{2}\left(\frac{d}{dt}\sum_{i,j}\left(\frac{\partial (u^t)^i}{\partial x^j}\right)^2_{|t=0}\right)\sqrt{\det({}^t du^0, du^0)} \\
&= \left(\sum_{i,j}\frac{\partial (u^0)^i}{\partial x^j}\frac{\partial X}{\partial x^j}\right)\sqrt{\det({}^t du^0, du^0)} \\
&= div_h(X)\sqrt{det(h)}
\end{aligned} \tag{4.10}$$

where $h = g_{|\Sigma}$ with $\Sigma = u^0(\mathbb{D})$. Which gives

$$\frac{d}{dt}\mathcal{A}(u^t)_{|t=0} = \int_\Sigma div(X)\, dv_h.$$

Let us split X as $X^T + X^\perp$ where X^T is the tangent part to Σ and $X^\perp$ the orthogonal one. Since $X^T \in C_c^\infty(\Sigma, T\Sigma)$, we get

$$\int_\Sigma div(X^T)\, dv_h = 0.$$

Moreover, let e_i an orthonormal frame of $T\Sigma$, we have

$$\begin{aligned} div_\Sigma(X^\perp) &= \sum_{i=1}^{2} h(\nabla_{e_i} X^\perp, e_i) \\ &= \sum_{i=1}^{2} e_i.h(X^\perp, e_i) - h(X^\perp, \nabla_{e_i} e_i) \\ &= -\sum_{i=1}^{2} X^\perp II(e_i, e_i)\vec{n} \\ &= -g(\vec{H}, X) \end{aligned} \tag{4.11}$$

where $\vec{H} = H\vec{n}$ Finally the first variation of the area is given by

$$\frac{d}{dt}\mathcal{A}(u^t)_{|t=0} = \int_\Sigma g(\vec{H}, X)\, dv_h.$$

Let us now compute the second variation, for this we consider a family of smooth immersion $u^t : \mathbb{D} \to N$ such that

$$\left.\frac{\partial u^t}{\partial t}\right|_{t=0} = \phi\vec{n} \text{ and with compact suppport.}$$

Setting $\mathcal{A}^t = \mathcal{A}(u^t)$, let x_i some local coordinates and we set $g_{i,j}(t) = g(u_i, u_j)$, we have

$$\mathcal{A}^t = \int_D \sqrt{det(g_{ij}(t))}$$

hence

$$\frac{\partial \mathcal{A}^t}{\partial t} = \frac{1}{2}\int_{\mathbb{D}} Tr(g^{ij}(t) g'_{ij}(t))\sqrt{det(g_{ij}(t))},$$

then

$$\frac{\partial^2 \mathcal{A}^t}{\partial t^2} = \frac{1}{2}\int_{\mathbb{D}} -Tr(g'_{ij}(t)g'_{ij}(t)) + Tr(g^{ij}(t)g''_{ij}(t)) + \frac{1}{2}\left(Tr(g^{ij}(t)g'_{ij}(t))\right)^2 \sqrt{det(g_{ij}(t))}.$$

But we know that

$$g'_{ij}(0) = g(u_{it}, u_j) + g(u_i, u_{jt}) = -2g(A_{ij}, u_t)$$

moreover since Σ is minimal, we finally have

$$\frac{\partial^2 \mathcal{A}^t}{\partial t^2}_{|t=0} = \frac{1}{2}\int_{\mathbb{D}} -4|II|^2|\phi|^2 + Tr(g^{ij}(0)g''_{ij}(0))dv_\Sigma.$$

Then,

$$\begin{aligned} Tr(g_{ij}(0)g''_{ij}(0)) &= 2\sum_i g''_{ii}(0) = \sum_i g(u_{itt}, u_i) + g(u_{it}, u_{it}) \\ &= 2\sum_i g''_{ii}(0) = \sum_i g(R(u_t, u_i)u_t, u_i) + div_\Sigma(u_{tt}) + g(u_{it}, u_{it}) \end{aligned}$$

since, ((2.14) of Druet)

$$g(u_{itt}, u_i) = g(u_{tit}, u_i) = g(u_{tti}, u_i) + g(R(u_i, u_t)u_t, u_i)$$

then

$$Tr(g_{ij}(0)g''_{ij}(0)) = 2\sum_i g(R(u_i, u_t)u_t, u_i) + div_\Sigma(u_{tt}) + g(u^T_{it}, u^T_{it}) + g(u^N_{it}, u^N_{it})$$

but

$$\sum_i g(u^N_{it}, u^N_{it}) = |\nabla\phi|^2$$

and

$$\sum_i g(u^T_{it}, u^T_{it}) = \sum_{ij} |g(II_{ij}, u_t)|^2$$

finally

$$Tr(g_{ij}(0)g''_{ij}(0)) = -2\phi^2 Ricc(\vec{n}, \vec{n}) + div_\sigma(u_{tt}) + 2\phi^2|II|^2 + 2|\nabla\phi|^2$$

then intégrating by parts,

$$\frac{d^2\mathcal{A}}{dt^2} = \int_\Sigma \phi\Delta\phi - \left(|II|^2 + Ric(\vec{n}, \vec{n})\right)\phi^2\, dv$$

Definition 4.3.1 *Let $u : \mathbb{D} \to \mathbb{R}^3$ be a smooth minimal immersion, $\Sigma = u(\mathbb{D})$, it is said stable if and only if for all $\phi \in C^\infty_c(\Sigma)$ we have*

$$\int_\Sigma |\nabla u|^2 - \left(|\mathrm{II}|^2 + Ric(\vec{n}, \vec{n})\right)\phi^2\, dv \geq 0$$

that is to say

$$\int_\Sigma |\nabla u|^2\, dv \geq \int_\Sigma \left(|\mathrm{II}|^2 + Ric(\vec{n}, \vec{n})\right)\phi^2\, dv.$$

We directly see that $Ricc \geq 0$ contradicts the existence of closed stable minimal surfaces.

Let us remind that the Gauss-equation for a surface Σ of a Riemannian manifold N gives

$$K = Rm_{1212} + \mathrm{II}_{11}\mathrm{II}_{22} - \mathrm{II}^2_{12}, \tag{4.12}$$

where the index are given by a fixed orthonormal frame (e_1, e_2) of $T\Sigma$. If we assume that Σ is minimal, then we have $\mathrm{II}_{11} + \mathrm{II}_{22} = 0$, which gives

$$K = Rm_{1212} - \frac{1}{2}|\mathrm{II}|^2.$$

Moreover,

$$\begin{aligned} R &= Ric_{11} + Ric_{22} + Ric_{33} \\ &= 2Rm_{1212} + Rm_{1313} + Rm_{2323} + Ric_{33} \\ &= 2Rm_{1212} + 2Ric_{33} \end{aligned} \tag{4.13}$$

Finally, thanks (4.12 and (4.13)

$$Ric_{33} = \frac{1}{2}R - \left(K + \frac{1}{2}|\mathrm{II}|^2\right) = \frac{R}{2} - K - \frac{1}{2}|\mathrm{II}|^2.$$

Hence a surface is stable if and only if for all $\phi \in C_c^\infty(\mathbb{D})$ we have

$$\int_\Sigma |\nabla\phi|^2 \, dv \geq \int_\Sigma \left(\frac{1}{2}|\mathrm{II}|^2 - K + \frac{R}{2}\right) \phi^2 \, dv. \tag{4.14}$$

Hence stability property is directly link to the sign of the scalar curvature. Here is two very nice applications of this remark.

Theorem 4.3.2 (Fischer-Colbrie, Schoen [11]) *Let (N, g) a complete Riemannian manifold with none-negative scalar curvature and Σ a complete minimal surface into N, then Σ is conformally equivalent to $\mathbb{C}$ or $S^1 \times \mathbb{R}$.*

Corollary 4.3.3 *The only complete stable minimal surface of $\mathbb{R}^3$ is the flat plane.*

Chapter 5

Proof of the positive mass theorem ($n = 3$)

Theorem 5.0.1 ([27]) *Let (N, g) an asymptotically flat manifold such that in the end*

$$g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij} + O_5(r^{-2}).$$

If $R_g \in L^1$ and $R_g \geq 0$ then $m \geq 0$.

Theorem 5.0.2 *Let (N, g) an asymptotically flat manifold with $g \in W^{2,q}(\mathbb{R}^3 \setminus B_1)$ with $q > 3$ and $\tau > \frac{n-2}{2}$. If $R_g \in L^1$ and $R_g \geq 0$ then $m \geq 0$.*

Theorem 5.0.3 *Let (N, g) an asymptotically flat manifold with $g \in W^{2,q}(\mathbb{R}^3 \setminus B_1)$ with $q > 3$ and $\tau > \frac{n-2}{2}$. If $R_g \in L^1$ and $R_g \geq 0$ and $m \geq 0$ the (N, g) is isometric to $(\mathbb{R}^3, \delta)$.*

Proof of 5.0.1:

We proceed by contradiction and assume that $m < 0$. We fix a chart at infinity $\mathbb{R}^3 \setminus B_1$.

Step 1: We can assume that $R_g > 0$ out side some compact K.

$$\begin{aligned}
-\Delta_g \left(\frac{1}{r}\right) &= -\frac{1}{\left(1 + \frac{m}{2r}\right)^6 r^2} \partial_i \left(\left(1 + \frac{m}{2r}\right)^2 r^2 \partial_i \left(\frac{1}{r}\right)\right) + O\left(\frac{1}{r^5}\right) \\
&= -\left(1 - \frac{3m}{r}\right) \frac{m}{r^4} + O\left(\frac{1}{r^5}\right) \\
&= -\frac{m}{r^4} + O\left(\frac{1}{r^5}\right) \\
&< 0 \text{ for } r \text{ big enough.}
\end{aligned} \tag{5.1}$$

Let $\chi_R \in C^\infty(\mathbb{R}_+, \mathbb{R})$ increasing and concave such that

$$\begin{cases} \chi_R(t) = t \text{ if } t \leq t_0 = \frac{-m}{8R} \\ \chi_R(t) = \frac{3t_0}{2} \text{ if } t \leq 2t_0. \end{cases}$$

Hence we define $\phi_R \in C^\infty(M)$ by

$$\begin{cases} \phi_R(x) = 1 + \frac{3t_0}{2} \text{ if } x \in K \\ \phi_R(x) = 1 + \chi_R\left(\frac{-m}{4|x|}\right) \text{ if } x \in \mathbb{R}^3 \setminus B_1. \end{cases}$$

then

$$-\Delta_g \phi_R = -g^{rr}\chi_R'' \left|\partial_r\left(\frac{-m}{4r}\right)\right|^2 - \chi_R' \Delta_g\left(\frac{-m}{4r}\right) < 0 \text{ on the whole } N \text{ for} R \text{ big enough.}$$

We fix R once for all. Let us make the following conformal change of metric $\tilde{g} = \phi_R^4 g$, then we remark that since

$$\begin{aligned} \tilde{g} &= \left(1 - \frac{m}{4}\right)\left(1 + \frac{2m}{2}\right)\delta + O_5\left(r^{-2}\right) \\ &= \left(1 + \frac{m/2}{2r}\right)\delta + O_5\left(r^{-2}\right) \end{aligned} \tag{5.2}$$

then $\tilde{g}$ is asymptotically flat with mass $\tilde{m} = \frac{m}{2} < 0$. Moreover

$$R_{\tilde{g}} = \phi^{-5}(-8\Delta_g\phi_R + R_g\phi_R) \geq 0 \text{ on } N \text{ and } > 0 \text{ for r large enough,}$$

which achieved the proof of Step 1.

Step 2: There exists Σ a complete stable properly immersed surface in N such that

$$\Sigma \cap K \textbf{ is compact}$$

and

$$\Sigma \cap (\mathbb{R}^3 \setminus B_1) \textbf{ is contained between two parallele planes}.$$

Let $C_R = \{(x^1, x^2, x^3) \,|\, (x^1)^2 + (x^2)^2 = R^2 \text{ and } x^3 = 0\} \subset \mathbb{R}^3 \setminus B_1$, considering the previous chapter we know that there exists a solution of the Plateau problem, more precisely a disc Σ_R that minimize the area among all disc that bound C_R.

Let $T_h = \{(x^1, x^2, x^3) \,|\, |x^3| \leq h\} \subset \mathbb{R}^3 \setminus B_1$.

Claim 2.1: There exists h_0 such that $\Sigma_R \subset B_{h_0}$ for all $R \geq 2$.

In order to prove this we are going to prove that ∂B_h is strictly mean-convex ($H > 0$) for h big enough, then the claim will be a consequence of the maximum principle. Indeed Let us consider a point of Σ_R where x^3 achieves its maximum (or minimum). If $|x^3 \leq 1$ there is nothing to prove, else it is an interior point and since

$$(\nabla^2 x^3)_{ij} = -\Gamma^3_{ij} = \frac{mx^j}{r^3}\delta_{i3} + \frac{mx^i}{r^3}\delta_{j3} - \frac{mx^3}{r^3}\delta_{ij} + O_4(r^{-4}),$$

then

$$\Delta_{\Sigma_R} x^3 = \frac{mx^3}{r^3} + O(r^{-4}) < 0 \text{ for } r \text{ big enough},$$

which contradicts the fact that the point is a maximum and proves the claim.

Claim 2.2: II_{Σ_R} is bounded on every compact, independently of R.

In fact we are going to show a bit more. Let us consider

$$s_R = \sup_{x \in \Sigma_R} \frac{d(C_R, x)|\mathrm{II}_{\Sigma_R}(x)}{1 + d(C_R, x)}|.$$

It is clear that if s_R is uniformly bounded we get the result. Else we consider a point x_R where the maximum is achieved. Of course we have

$$\mathrm{II}_{\Sigma_R}(x_R) \to +\infty \text{ as } R \to +\infty.$$

We Consider the ball $B(x_R, d_R)$ where $d_R = \min(\delta/2, d(x_R, C_R))$, where $\delta > 0$ is a lower bound on the injectivity radius of N. Then we dilate it by a factor $\mu_R = |\mathrm{II}_{\Sigma_R}(x_R)|$. Then we obtain a surface $\tilde{\Sigma}_R = \mu_R(\Sigma_R - x_R)$ in a ball $(B(0, D_R), g_R)$ where $D_R \to +\infty$ and $g_R \to \delta$ on every compact. Moreover

$$\begin{aligned} |\mathrm{II}_{\tilde{\Sigma}_R}(x)| &= \frac{1}{\mu_R}\left|\mathrm{II}_{\Sigma_R}\left(x_R + \frac{x}{\mu_R}\right)\right| \\ &\leq \frac{1 + d\left(x_R + \frac{x}{\mu_R}, C_R\right)}{d\left(x_R + \frac{x}{\mu_R}, C_R\right)} \frac{d(x_R, C_R)}{d(x_R, C_R) + 1} \\ &\leq 2, \end{aligned} \tag{5.3}$$

in order to get the last inequality we used the fact that $\mu_R d(x_R, C_R) \to +\infty$. Moreover

$$|\mathrm{II}_{\tilde{\Sigma}_R}(0)| = 1.$$

Then we apply the following theorem

Theorem 5.0.4 (see theorem [?]) *Let Σ_n a sequence of smooth minimal surfaces in (N, g) a smooth complete Riemmannian manifold with bounded geometry*[1]*, let $p \in N$ and $\delta > 0$ the injectivity radius at p such that*

[1] i.e. with bounded curvature and injectivity radius bounded from below.

- *there exists* $p_p \in \Sigma_n$ *such that* $p_n \to p$,
- $\sup |B(p,\delta) \cap \Sigma_n| < +\infty$,
- $\sup\limits_{\Sigma_n \cap B(p,\delta)} |\mathrm{II}_{\Sigma_n}| < +\infty$.

Then We can extract a subsequence of Σ_n *that converge* C^2 *in* $B\left(p, \frac{\delta}{2}\right)$.

This theorem is a consequence of the fact that we can see Σ_n as a graph above T_pN and that it satisfies some elliptic P.D.E..

Since the geometry of $(B(0, D_R), g_R)$ is bounded, using the fact that the area is bounded on any compact which can be insured by selecting only one connected component of $\Sigma_R \cap (B(0, D_R)$ and applying the comparison principle of Step 3, then thanks to some diagonal extraction principle, we can extract a subsequence of $\tilde{\Sigma}_R$ that converges to $\tilde{\Sigma}_\infty$ a complete stable minimal surface of $\mathbb{R}^3$. The theorem 4.3.2 insures that it is a plane but by C^2 convergence we must have $|\mathrm{II}_{\Sigma_\infty}(0)| = 1$, which leads to a contradiction and prove Claim 2.2.

Then to prove step 2 it suffices to check that Σ_R satisfies the hypothesis of theorem 5.0.4. The fact that the minimal surface is contained between two planes insures that it can't escape to infinity. The second hypothesis is satisfies thanks to the comparison principle of Step 3 which adapted to this situation insure that for any compact K there exists $C_K > 0$ such that for every R we get

$$|\Sigma_R \cap K| \leq C_K.$$

The third hypothesis is insured by Claim 2.2., which achieves the proof of Step 2.

Step 3: Let, for $R > 1$, $S_R = \Sigma \cap (B_R \setminus B_1)$ then

$$\mathcal{A}(S_R) = O(R^2).$$

Moreover, for $a > 2$, we have

$$\int_{\Sigma \setminus B_1} \frac{1}{1 + r^a}\, d\sigma < +\infty,$$

and there exists $C > 0$ such that for all $R_2 \geq R_1 > 1$, we get

$$\int_{S_{R_2} \setminus S_{R_1}} \frac{1}{r^2}\, d\sigma < C \ln\left(\frac{R_2}{R_1}\right).$$

By transversality, for almost every R, ∂B_R is transverse to Σ. Hence $\Sigma \cap \partial B_R$ is a union of C^2 curves. Then using that for R large enough $\Sigma \setminus B_R$ is close to the plane $x^3 = 0$ (simple adaptation of Claim 2.1) and the curvature estimate, we deduce that, for R large enough, $\Sigma \cap \partial B_R$ is a simple connected curve, hence it separates ∂B_R on two hemisphere, any of which can be used as comparison surface, see figure 5.1, since for R large enough, the metric is almost flat and their area does not exceed $2 * (4\pi R^2)$, which proves the first estimate.

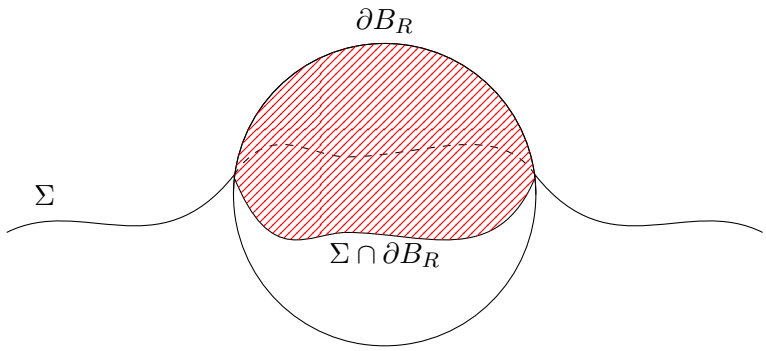

Figure 5.1: Comparison surface

Let $a > 2$, then

$$\begin{aligned}\int_{\Sigma\setminus B_1} \frac{1}{1+r^a}\, d\sigma &= \int_1^\infty \frac{d}{dt}\int_{S_t} \frac{1}{1+r^a}\, d\sigma \\ &= \int_1^\infty \frac{1}{1+t^a}\frac{d\mathcal{A}(S_t)}{dt}\, d\sigma \\ &= \int_1^\infty \frac{at^{a-1}}{(1+t^a)^2}\mathcal{A}(S_t)\, d\sigma \\ &= O(1),\end{aligned} \tag{5.4}$$

where we use in the last estimate that $a > 2$ and $\mathcal{A}(S_t) = O(t^2)$. The last estimate is obtained by a similar reasoning.

Step 4 :

$$\int_\Sigma K\, d\sigma > 0.$$

First we are going to prove that K is integrable. Since Σ a stable minimal surface for any $f \in C_c^\infty(\Sigma)$ we have

$$\int_\Sigma (|\mathrm{II}|^2 + Ric(\vec{n},\vec{n}))f^2\, d\sigma \leq \int_\Sigma |\nabla f|^2\, d\sigma.$$

Let us consider $f_R \in C_c^\infty(\Sigma)$ defined as below

$$f_R(x) = \begin{cases} 1 & \text{if } x \in \Sigma_R \cup K \\ \frac{\ln\left(\frac{R^2}{|x|}\right)}{\ln(R)} & \text{if } x \in \Sigma_{R^2} \cup \Sigma_R \\ 0 & \text{else.} \end{cases}$$

Then

$$\int_{\Sigma_R} |\mathrm{II}|^2 \, d\sigma \leq \int_\Sigma |\nabla f_R|^2 \, d\sigma + \int_\Sigma |Ric| f^2 \, d\sigma.$$

But it is trivial that

$$|Ric| = O(r^{-3})$$

and, thanks to the previous step,

$$\int_{\Sigma_{R^2} \setminus \Sigma_R} |\nabla f_R|^2 \, d\sigma \leq \int_{\Sigma_{R^2} \setminus \Sigma_R} \frac{1}{\ln(R)^2 r^2} \, d\sigma = O\left(\frac{1}{\ln(R)}\right).$$

Which prove that $|\mathrm{II}|^2$ is integrable, and then also $|K| \leq |Ric| + |\mathrm{II}|^2$. Now using (4.14) we get

$$\int_\Sigma \left(\frac{R_g}{2} - K + \frac{|\mathrm{II}|^2}{2}\right) d\sigma \leq 0,$$

finally using the fact that $R_g > 0$ at infinity, we conclude.

Step 5: Contradiction

Our goal is to prove that in fact

$$\int_\Sigma K \, d\sigma \leq 0.$$

In order to do so, let $C_R = \{(x^1, x^2, x^3) \,|\, (x^1)^2 + (x^2)^2 = R^2\}$, it suffices to prove that for R large enough

$$\int_{\Sigma \cap C_R} k_g \, dl \geq 2\pi + o(1),$$

then the Gauss-Bonnet formula will gives the conclusion.

Let us denote $\Gamma_R = \Sigma \cap C_R$, we choose a orhtonormal basis (e_1, e_2, e_3) in a neighborhood of Γ_R such that e_1 is tangent to Γ_R, e_2 point to the interior of the cylinder and e_3 is normal to Σ. We have

$$k_g = g(\nabla_{e_1} e_1, e_2).$$

let us denote $r' = \sqrt{(x^1)^2 + (x^2)^2}$, we remark that

$$g(\nabla r', e_1) = 0,$$

then, by our decreasing assumption on g,

$$g(\nabla_{e_1} e_1, \nabla r') + g(e_1, \nabla_{e_1} \nabla r') = 0,$$

but

$$\nabla r' = \frac{(x, y, 0)}{r} + O(r^{-1})$$

and

$$\nabla_{e_1} \nabla r' = \frac{e_1}{r} - \frac{1}{r} \left\langle e_1, \frac{\partial}{\partial x^3} \right\rangle \frac{\partial}{\partial x^3} + O\left(\frac{1}{r^2}\right)$$

which gives

$$\left\langle \nabla_{e_1} e_1, \frac{(x, y, 0)}{r} \right\rangle + \frac{1}{r} - \frac{1}{r} \left\langle e_1, \frac{\partial}{\partial x^3} \right\rangle^2 = O(r^{-1}) \|\nabla_{e_1} e_1\| + O(r^{-2}).$$

Finally using the fact that

$$\nabla_{e_1} e_1 = k_g e_2 - \mathrm{II}_{11} \vec{n} \tag{5.5}$$

we get

$$k_g \left\langle e_2, \frac{x'}{r} \right\rangle + \frac{1}{r} - \mathrm{II}_{11} \left\langle \vec{n}, \frac{(x, y, 0)}{r} \right\rangle - \frac{1}{r} \left\langle e_1, \frac{\partial}{\partial x^3} \right\rangle^2 = O(r^{-1}) \|\nabla_{e_1} e_1\| + O(r^{-2}).$$

Then

$$\begin{aligned} \int_{\Sigma \cap C_R} k_g \, dl &= \int_{\Sigma \cap C_R} k_g \left(1 + \left\langle e_2, \frac{x'}{r} \right\rangle\right) dl + \int_{\Sigma \cap C_R} \frac{1}{r} \, dl - \int_{\Sigma \cap C_R} \mathrm{II}_{11} \left\langle \vec{n}, \frac{(x, y, 0)}{r} \right\rangle dl \\ &\quad - \int_{\Sigma \cap C_R} \frac{1}{r} \left\langle e_1, \frac{\partial}{\partial x^3} \right\rangle^2 dl + \int_{\Sigma \cap C_R} O(r^{-1}) \|\nabla_{e_1} e_1\| \, dl + \int_{\Sigma \cap C_R} O(r^{-2}) \, dl \\ &= A + B + C + D + E + F. \end{aligned} \tag{5.6}$$

The extimate is going to follow from the following claim:

Claim: For R large enough $\Sigma \setminus B_R$ is a graph over $\mathbb{R}^2 \times \{0\}$, $z = u(x, y)$, with $C > 0$ such that

$$|\nabla u| \leq \frac{1}{|x'|},$$

and

$$|\nabla^2 u| \leq \frac{1}{|x'|^2}.$$

Then we easily get that A, C, D, E and F goes to zero ans since $B \geq \frac{length(\Sigma \cap C_R)}{R} \leq 2\pi$ this conclude the proof.

Proof of the CLaim, to be done or look at [26]

□

In order to prove theorem 5.0.2 and 5.0.3, we need the following result which will permit us to keep the scalar curvature none-negative through perturbations.

Theorem 5.0.5 *There exists $\varepsilon > 0$ such that for any (N, g) asymptotically flat manifold such that*

$$\int_N (R_-)^{\frac{3}{2}}\, dv \leq \varepsilon$$

where $R_- = \max\{0, -R\}$ is the negative part of R_g, then there exists a unique $u > 0$ such that $L_g(u) = 0$ and $u \to 1$ at infinity.

Proof of theorem 5.0.5:
In order to solves $L_g(u) = 0$, it suffies to solves $L_g(v) = -c(n)R_g$ and to set $u = v+1$. But for $R_g = 0$, we know from theorem **??**, that $L_g = \Delta_g$ is an isomorphism of $W^{2,q}_{-\tau}$ onto $L^q_{-\tau}$ for $\delta \in \left(\frac{n-2}{2}, n-2\right)$. In particular the first eigenvalue is strictly positive, then we have some room, if the scalar curvature become not too negative then this property remains true and the operator L_g is still invertible. Which prove the existence and uniqueness of v and then of u. Let us prove that u is onoe-negative. By contradiction if it were false we will consider the compact domain $\Omega = \{x \,|\, u(x) < 0\}$, on

$$\begin{aligned} 0 = \int_\Omega u L_g(u)\, dv = \int_\Omega |\nabla u|^2 + R_g u^2\, dv &\geq \frac{1}{C}\left(\int u^6\, dv\right)^{\frac{1}{3}} - c(n)\left(\int u^6\, dv\right)^{\frac{1}{3}}\left(\int_\Omega R_g^{\frac{3}{2}}\right)^{\frac{2}{3}} \\ &\geq \left(\frac{1}{C} - \varepsilon_0 c(n)\right)\left(\int u^6\, dv\right)^{\frac{1}{3}} \end{aligned} \tag{5.7}$$

where C is here the best constant for the Sobolev injection $W^{1,2}_0(\Omega) \subset L^6(\Omega)$. Of course for ε_0 small enough, we immediately get $u \equiv 0$ and the desired contradiction. Finally u is in fact positive by maximum principle, which achives the proof of the theorem. □

Proof of theorem 5.0.3:

First of all we remark that $R_g \equiv 0$, else making a pertubation like in Step 1 of the proof of theorem 5.0.1, we could decrease slightly the scalar curvature keeping it non-negative and making the mass negative, which is going to contradict theorem [**?**].

Then Let us consider $\tilde{g}_t = u_t^4(g + t\eta Ric)$ where η is any cut-off function and u_t is given by theorem 5.0.5. Then, thanks to theorem 5.0.1, $m(g_t)$ achieves its minimum at $t = 0$. Then

$$0 = \left.\frac{dm(g_t)}{dt}\right|_{t=0}.$$

But

$$\left.\frac{dg_t}{dt}\right|_{t=0} = 4vg + h,$$

where $v = \left.\frac{u_t}{dt}\right|_{t=0}$ and $h = \eta Ric$, which gives

$$\begin{aligned}\left.\frac{d}{dt}\left(\int_{\partial B_R} \partial_k g_{ik} - \partial_i g_{kk}\nu^i \, dv\right)\right|_{t=0} &= \left.\frac{d}{dt}\left(\int_{B_R} div(-\nabla tr(h) + div(h)) \, dv\right)\right|_{t=0} \\ &= \left.\frac{d}{dt}\left(\int_{B_R} R_{\tilde{g}_t} \, dv\right)\right|_{t=0} + \int_{B_R} h^{ij} G_{ij} \, dv + o(1).\end{aligned} \tag{5.8}$$

where $M_R = K \cup B_R \setminus B_1$, G is the Einstein tensor. Here we use (1.22) and (1.23). Finally, passing to the limit when $R \to +\infty$ and using that $R_{\tilde{g}_t} \equiv R_g \equiv 0$, we get that

$$0 = \left.\frac{dm(\tilde{g}_t)}{dt}\right|_{t=0} = \int_N \eta |Ric|^2 \, dv,$$

which proves the theorem. □

Solutions to exercises

Answer of exercise 1.1

This is a solution of Ex 1

Answer of exercise 1.2

This is a solution of Ex 2

Answer of exercise 1.3

This is a solution of Ex 3

Answer of exercise 1.4

This is a solution of Ex 4

Answer of exercise 1.5

This is a solution of Ex 5

Answer of exercise 1.6

This is a solution of Ex 6

Answer of exercise 1.8

This is a solution of Ex 7

Answer of exercise 1.8

This is a solution of Ex 7

Answer of exercise 1.9

This is a solution of Ex 8

Answer of exercise 1.9

This is a solution of Ex 8

Answer of exercise 1.11

This is a solution of Ex 11

Answer of exercise 1.12

This is a solution of Ex 12

Answer of exercise 1.13

This is a solution of Ex 13

Answer of exercise 1.14

This is a solution of Ex 14

Answer of exercise 2.1

This is a solution of Ex 21

Answer of exercise 2.2

This is a solution of Ex 22

Answer of exercise 2.3

This is a solution of Ex 23

Answer of exercise 2.4

This is a solution of Ex 24

Answer of exercise 2.5

This is a solution of Ex 25

Answer of exercise 2.6

This is a solution of Ex 26

Answer of exercise 2.7

This is a solution of Ex 27

Answer of exercise 2.8

This is a solution of Ex 28

Answer of exercise 2.9

This is a solution of Ex 29

Answer of exercise 3.1

This is a solution of Ex 31

Answer of exercise 3.2

This is a solution of Ex 32

Answer of exercise 3.3

This is a solution of Ex 33

Answer of exercise 3.4

This is a solution of Ex 34

Answer of exercise 3.4

This is a solution of Ex 34

Answer of exercise 4.1

This is a solution of Ex 41

Answer of exercise 4.2

This is a solution of Ex 42

Answer of exercise 4.3

This is a solution of Ex 43

Answer of exercise 4.4

This is a solution of Ex 44

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Index