UNIVERSITÉ DE PARIS

HABILITATION À DIRIGER DES RECHERCHES

Sur l'analyse de quelques problèmes invariants conformes

Auteur :
Paul LAURAIN

Composition du Jury :

Fernando Coda Marques	Princeton University	Rapporteur et examinateur
Frédéric Hélein	Université de Paris	Rapporteur interne et examinateur
Karen Uhlenbeck	Texas University	Rapporteure
Jean Dolbeault	Université de Dauphine	Examinateur
Olivier Druet	Université de Lyon	Examinateur
Tristan Rivière	ETH Zürich	Examinateur
Peter Topping	Warwick University	Examinateur

Mémoire présenté en vue de l'obtention du Diplôme d'Habilitation à Diriger des Recherches de l'Université de Paris

4 Décembre 2019

« *J'aimais et j'aime encore les mathématiques pour elles-mêmes comme n'admettant pas l'hypocrisie et le vague, mes deux bêtes d'aversion.* »

Henri Beyle, dit Stendhal

Résumé :

Le but de ce mémoire est de présenter les résultats que j'ai obtenus depuis la fin de ma thèse sur l'analyse des problèmes invariants conformes. Il s'agit de problèmes tels que les équations de type-Yamabe, les applications harmoniques et bi-harmoniques, les surfaces minimales, les surfaces de Willmore qui partagent tous une structure analytique particulière. Mon travail s'est basé sur cette structure pour démontrer des résultats tels que la régularité, la stabilité, la compacité des suites de solutions ou encore la convexité de leurs fonctionnelles.

Abstract :

The purpose of this thesis is to present the results I have obtained since the end of my thesis on the analysis of conformally invariant problems. These are problems such as Yamabe-type equations, harmonic and bi-harmonic maps, minimal surfaces and Willmore surfaces that all share a particular analytic structure. My work has consisted to use this structure to prove results such as regularity, stability, compactness of the sequences of solutions or the convexity of their functionals.

A la mémoire de mon père,

Remerciements

Fernando Coda Marques, Frédéric Hélein et Karen Uhlenbeck ont accepté d'être rapporteurs de cette habilitation. Je leur exprime ici ma gratitude. Je tiens également à remercier sincèrement les membres du jury : Fernando Coda Marques, Jean Dolbeault, Frédéric Hélein, Olivier Druet, Tristan Rivière et Peter Topping.

Depuis la fin de ma thèse, j'ai effectué la plus grande partie de mes recherches à l'IMJ. Je tiens à remercier l'ensemble de mes collègues pour les discussions stimulantes et également les moments de détente. Je n'oublie pas nos gestionnaires qui n'ont pas manqué de m'épauler dans mes taches administratives.

Je tiens également à remercier Rafe Mazzeo et le département de mathématiques de l'Université de Stanford pour m'avoir accueilli avec ma famille pendant 8 mois. Ce séjour m'a beaucoup apporté tant sur le plan mathématiques que sur le plan humain.

Je voudrais remercier particulièrement Nicolas Marque, Alexis Michelat et Gabriel Rivière pour leur relecture de ce manuscrit et leurs remarques pertinentes.

Enfin je ne n'ai pas de mot assez fort pour remercier celle qui m'accompagne depuis le début et sans qui tout ceci aurait peu de sens. Merci Anne !

Table des matières

Chapitre 1

Introduction

1.1 Version française

L'objet de ce mémoire est de présenter les différents travaux que j'ai effectués depuis ma thèse et de donner quelques perspectives quant à mes recherches futures.

Dans ma thèse j'avais en particulier développé une analyse ponctuelle afin d'étudier les phénomènes de *blow-up* pour l'équation des surfaces à courbure moyenne constante (CMC). Par la suite, je me suis intéressé à un ensemble plus vastes d'équations : les problèmes variationnels invariants par transformation conforme. Si je devais définir mes travaux aujourd'hui en quelques mots, je dirais que je fais de l'analyse de *blow-up* pour les problèmes conformément invariants. L'invariance conforme joue un rôle fondamental dans un grand nombre de problèmes issus de la physique (relativité générale, théorie conforme des champs, turbulence...), et de la géométrie (géométrie conforme, surfaces de Riemann, théorie de Yang-Mills,...). Comme nous le verrons, notamment au chapitre 3, cette invariance par transformations conformes offre une structure particulière aux équations donnant naissance à des phénomènes tels que la compacité par compensation. Toutefois, elle est également à l'origine de difficultés analytiques intrinsèques quant à l'étude des équations qui régissent ces problèmes, comme leur caractère critique et les phénomènes de *blow-up*. Toute la difficulté et la beauté de ces équations tient à ce qu'à la fois les problèmes qu'offrent leur étude mais aussi leurs solutions sont liés à leur essence même.

L'exemple le plus élémentaire d'un tel problème est celui des applications harmoniques. Il s'agit de la généralisation naturelle du problème des géodésiques. Il est considéré comme l'exemple fondamental en calcul des variations, dont je me permets de rappeler ici la fonctionnelle

$$E(u) = \int_{\Sigma} |\nabla u|_g^2 \, dv_g,$$

où u prend ces valeurs dans une variété N fixées. Si on considère N comme une sous-variété de $\mathbb{R}^m$, l'équation d'Euler-Lagrange s'écrie alors comme suit

$$\Delta u \perp T_u N.$$

Malgré la simplicité de la formulation de ce problème, l'étude de la régularité et du comportement des suites de solutions s'est avérée un problème très ardu. Pour la régularité en dimension 2 (dimension de l'invariance conforme), il a fallu attendre les travaux de Hélein au début des années 90 et la première analyse de *blow-up* a été faite par Parker au milieu de cette même décennie. Ces avancés fondamentales ne peuvent être rappeler ici, sans évoquer les travaux précurseurs de Wente en 69 puis de Brezis-Coron en 82, pour l'équation des CMC :

$$\Delta u = -2u_x \wedge u_y.$$

Cette équation est également centrale en analyse géométrique puisqu'elle trouve son origine dans le problème isopérimétrique. Les questions centrales pour ces problèmes sont, après celles d'existence, d'unicité et de régularité, les questions de compacité des suites de solutions et, lorsque celle-ci n'a pas lieu, l'analyse des phénomènes de *blow-up*, d'un point vu énergétique tout d'abord et enfin d'un point de vue ponctuel. Mes recherches ont donc été guidées par ce souci d'explorer les propriétés fines d'objets fondamentaux comme les applications harmoniques, les surfaces CMC mais aussi d'autres variantes d'analyse géométrique comme les surfaces minimales ou les surfaces de Willmore et plus généralement de la grande variété des objets qui s'inscrivent dans le cadre à la fois riche et complexe de l'invariance conforme en calcul des variations.

Pour ce mémoire j'ai divisé mon travail en 5 parties, mais le lecteur remarquera aisément que tous ces travaux sont fortement reliés.

Le chapitre 2 est dédié à l'identité de Pohožaev. Ce choix n'est pas anodin, même si cela correspond à mes travaux les plus anciens. En effet on va retrouver cette identité, ainsi que ses nombreuses variantes, tout au long du mémoire comme un argument essentiel dans l'analyse des problèmes conformément invariants. D'ailleurs on peut déjà noter son utilisation par Uhlenbeck dans son travail fondateur sur le *point-removability* [**ScU**, **SU**]. Je développe également dans ce mémoire une généralisation de cette idée qui repose sur le lien étroit entre l'identité de Pohožaev et les solutions de l'équation linéarisée du problème non-linéaire, plus particulièrement en les réinterprétant à travers le théorème de Noether. Cette idée est développée dans la section 5.1 et utilisée dans les sections suivantes pour combler l'absence d'identité de Pohožaev (ou de manière équivalente de formule de flux) pour l'équation de Willmore.
Ce chapitre, commence donc par une brève introduction à l'identité de Pohožaev "classique" et l'obstruction qui en découle pour les équations de type Yamabe, dont je rappelle ici la forme générale

$$\Delta u + hu = u^{2^*-1},$$

où 2^* est l'exposant critique pour l'injection de Sobolev de H^1. La criticité de l'exposant la rend naturellement sujette à des phénomènes de concentration. L'analyse de ces phénomènes a été un enjeu majeur dans l'étude de l'espace des métriques à courbure scalaire constante, voir [**BM**]. Cela nous permet également d'introduire la stratégie d'analyse de *blow-up* proposée par Schoen pour cette équation scalaire et qui sera une base de travail pour les equations vectorielles. Dans les section 2.2 et 2.3, je présente les résultats, obtenus avec Druet puis Druet et Hebey, sur la stabilité de l'obstruction de Pohožaev, question initialement soulevée par Brezis et Willem [**BW**]. Nous montrons que cette obstruction persiste par rapport à des perturbations $C^{0,\eta}$ de la linéarité et $C^{1,\eta}$ du domaine. De plus, ce résultat est optimal puisque nous

montrons également l'existence de solutions pour des perturbations dans des espaces plus gros.
Enfin je profite de ce chapitre pour présenter, dans la section 2.4, une identité de Pohožaev pour les applications 1/2-harmoniques, obtenue avec Da Lio et Rivière. Dans le cas des applications harmoniques, l'identité vient lier les parties angulaires et radiales du gradient. Pour les applications 1/2-harmoniques, ce sont les parties paires et impaires qui sont liées.

Le chapitre 3 est dédié au traitement des phénomènes de quantification de l'énergie pour différents problèmes. Pour simplifier, disons qu'il s'agit de montrer que la limite des énergies est égale à la somme des énergies des limites, ce qui impose naturellement une quantification de l'énergie lorsque les solutions limites sont classifiées. Au-delà de la résolution de questions ouvertes comme la quantification des applications bi-harmoniques, j'y présente une stratégie très robuste, développée avec Rivière, et qui a fait ses preuves sur beaucoup de problèmes sujets à une invariance conforme.
Tout d'abord dans les quatre premières sections j'introduis la classe des problèmes invariants conformes, le concept précis de quantification (ou d'identité d'énergie) que nous explorons à travers des exemples concrets.
Puis dans les sections 3.5 et 3.6, j'expose les grandes lignes de notre premier résultat avec Rivière, à savoir que toute suite de solutions d'un problème invariant conforme en dimension 2 vérifie une identité d'énergie. Ce résultat d'une très grande généralité relève d'une stratégie qui sera par la suite appliquée dans de nombreuses de situations similaires.
Les ingrédients de la preuve reposent sur trois piliers : les lois de conservations découvertes par Rivière pour les équations elliptiques à potentiel antisymétrique [**Ri07**], les espaces de Lorentz (avec, comme clef de voûte, une généralisation du lemme de Wente sur des anneaux dont la classe conforme dégénère), et enfin l'utilisation de l'identité de Pohožaev.
Dans les dernières sections, j'expose des résultats de quantifications pour les applications bi-harmoniques, les applications harmoniques à bord libre et enfin leurs jumelles les applications 1/2-harmoniques.

Le chapitre 4 présente une étude des suites de solutions de problèmes invariants conformes lorsque la classe conforme de la surface sous-jacente dégénère. Après avoir présenté la décomposition de Deligne-Munford qui nous assure que l'on peut toujours trouver un atlas conforme constitué d'un nombre fini de disques et de *collars*. Nous présentons deux résultats de quantification en l'absence de contrôle de la classe conforme. Le premier théorème est une amélioration du résultat de compacité de Gromov pour les courbes J-holomorphes, le second concerne les applications harmoniques vérifiant une condition cohomologique particulière. En effet il n'y a pas de quantification dans le cas général pour les applications harmoniques comme l'avait déjà observé Parker [**Pa**].
Dans un second temps nous nous intéressons au contrôle du facteur conforme pour les surfaces dont la seconde forme fondamentale est bornée dans L^2. Ce résultat vient généraliser les travaux fondateur de Hélein et Müller-Šverák [**He**, **MuSv**]. Ces derniers ont traité le cas du disque, nous montrons que l'on peut également contrôler le facteur conforme dans les *collars*. Pour cela nous démontrons, avec Rivière, un contrôle uniforme de la fonction de Green du laplacien dans l'espace L^2-faible indépendant de la classe conforme. Nous pensons que ce résultat d'une grande généralité pourrait

trouver des applications au delà du champ de l'analyse géométrique, étant donné le role fondamentale joué par la fonction de Green des surfaces de Riemann dans bon nombre de domaines.
Dans la section suivante, je présente comment en m'appuyant sur ce dernier résultat j'ai démontré, avec Rivière, la quantification pour les suites de surfaces de Willmore dont la classe conforme dégénère. Comme pour les applications harmoniques, sans hypothèse additionnelle, il peut y avoir des pertes d'énergie dans les *collars*. Toutefois nous quantifions cette perte par un résidu. Cette dernière quantité semble cruciale, en particulier nous montrons qu'elle permet de distinguer les surfaces de Willmore conformément minimales de celles qui ne le sont pas. Le théorème de Noether joue encore un rôle prépondérant dans cette dernière remarque puisque ce résidu trouve son origine dans la loi de conservation donnée par l'invariance par rotation. Je présente également des résultats de compacité (lorsque la classe conforme est bornée), obtenu dans le même article, ceux-ci étant obtenu à l'aide du résidu donné par l'invariance par translation.
Enfin je décris un travail en préparation avec Lira, dont le but est de réaliser les premiers *gluing* de surfaces de Willmore, avec comme objectif final la production d'exemples explicites de surfaces de Willmore d'énergie bornée dont la classe conforme dégénère.

Le chapitre 5 a pour but but de présenter le comportement des suites de surfaces CMC, mais surtout Willmore, dans un cadre riemannien. Plus précisément nous considérons la situation de petites surfaces qui se concentrent autour d'un point et de grandes surfaces dans des espaces asymptotiquement euclidien. D'une part la présence de courbure rend ces surfaces moins rigides puisque que l'on peut, par exemple, coller deux sphères rondes en une CMC comme l'a démontré Zolotareava [**Zo**]. D'autre part l'espace n'étant plus homogène, la localisation de ces surfaces est contrainte, comme je l'avais déjà montré dans ma thèse pour les CMC.
Au début du chapitre j'explique comment surmonter ces difficultés via une étude fine du linéarisé. Dans un premier temps, il faut obtenir des estimées précises malgré l'absence de principe du maximum. Puis toujours à travers le linéarisé et à l'aide des estimées obtenues on peut observer comment la géométrie contraint la localisation de ces objets. La technique employée ici vient généraliser en quelque sorte l'identité de Pohožaev ou de manière équivalente la formule du flux.
Le premier résultat, obtenu avec Mondino, consiste à montrer que les surfaces de Willmore d'énergie plus petite que 8π se concentrent nécessairement autour d'un point critique de la courbure scalaire. Nous étendons également ce résultat aux surfaces Willmore d'aire contrainte, ce qui nous donne un corollaire intéressant concernant la masse de Hawking.
Ce lien entre entre les surfaces de Willmore et la relativité générale est développé dans la dernière section. J'y détaille un programme, que je mène en collaboration avec Metzger, dont le but ultime est d'obtenir une généralisation de la masse de Hawking. J'expose des résultats intermédiaires sur l'unicité des feuilletages CMC et Willmore, dont l'existence a été obtenue respectivement par Huisken-Yau [**HuY**] et Lamm-Metzger-Schultze [**LMS1**].

Le chapitre 6 traite du problème de la convexité de certaines fonctionnelles invariantes conforme. En effet il est bien connu que dès que la variété d'arrivée est courbée positivement, on perd l'unicité des applications harmoniques. Toutefois, comme l'ont démontré Colding et Minicozzi dans [**CM08**], on peut retrouver de la

convexité et donc de l'unicité autour des points critiques de petites énergies.
Dans une première section, je donne une preuve de ce résultat qui repose uniquement sur l'ε-régularité. Cette preuve très robuste a déjà trouvé deux applications à d'autres problèmes invariants conformes que j'expose dans les sections suivantes.
Tout d'abord, avec Petrides, nous avons démontré une convexité locale pour les applications harmoniques à bord libre, ce qui nous a permis de developer une procédure de remplacement dans ce cadre et d'obtenir existence de surfaces minimales à bord libre réalisant le *width* d'une variété à bord.
Enfin, avec Lin, nous avons démontré cette convexité de l'energie dans le cas des applications bi-harmoniques à valeurs dans la sphère, ce qui ouvre la porte au développement de procédure de remplacement et surtout de min-max en dimension 4.

Ce mémoire ne reflète que la partie publiée ou en passe de l'être de mes travaux. Mais j'aimerais cependant aborder en guise de conclusion de l'introduction d'autres axes de ma recherche sur lequel je travaille activement mais qui à ce jour ne peuvent bénéficier d'un chapitre à part entière.

— Avec Petrache, nous venons de pré-publier un résultat sur la rigidité des mesures unidimensionnelles uniformément distribuées en toute codimension. En fait il s'agit d'un travail préparatoire au cas des surfaces qui est une question assez ouverte. Pour l'instant il n'y a pas d'argument clair dans le sens d'une rigidité. Cette question de théorie géométrique de la mesure est en fait très liée à mes travaux, puisque le problème est dans un certain sens invariant conforme et que de plus sa résolution passe par le choix d'une jauge adaptée de type Coulomb.

— Une autre question sur laquelle je travail activement est l'étude des applications harmoniques à valeur dans un espace Lorentzien. En effet comme je l'explique à la fin de 3.3, cette question est largement ouverte. Toutefois, ces derniers mois j'ai réalisé des avancées significatives, ce qui pourrait notamment nous aider dans la compréhension des surfaces de Willmore. En effet, La correspondence de Bryant nous assure que l'on peut associer à toute surface de Willmore une surface minimale dans l'espace de Sitter et par conséquent une application harmonique.

— Comme expliqué par Alexakis et Mazzeo dans [**AM2**], les surfaces minimales dans l'espace hyperbolique peuvent s'interpréter comme un cas particulier de surfaces de Willmore (euclidienne) à bord libre. Toutefois l' ε-régularité au bord, ne semble pas complètement résolu. Les techniques présenté dans ce mémoire pour Willmore mais aussi pour les problème à bord libre sont de bons candidats pour résoudre complètement à cette question. Le problème semble toutefois très ardu car même la question de la régularité optimale (c'est à dire C^1) du bord dans l'infinie conforme, déjà soulevé dans [**deO**], reste ouverte à ce jour.

— Précédemment dans [**AM1**], Alexakis et Mazzeo avaient motivé leur étude des surfaces minimales dans l'espace hyperbolique, en soulignant l'existence d'un dictionnaire avec le problème des métriques de Poincaré-Einstein en dimension 4, que l'on peut résumer brièvement dans le tableau suivant

Dimension 2	dimension 4
Surfaces minimales dans $\mathbb{H}^3$	Métriques Poincaré-Einstein
Aire renormalisée	Volume renormalisé
$\int \lvert \mathring{A} \rvert^2$	$\int \lvert W \rvert^2$
Surfaces de Willmore dans $\mathbb{R}^3_+$	Métriques Bach

Comme le montre le papier récent de Chang et Ge **[ChGe]**, là aussi l'ε-régularité au bord n'est pas complètement comprise. Bien que le problème soit intrinsèque, contrairement à ceux étudiés dans ce mémoire, comme il jouit d'une invariance conforme au travers de l'énergie de Weyl, il devrait aussi être sensible aux techniques que j'ai développées.

Les sujets évoqués ci-dessus sont extrêmement actifs aujourd'hui, probablement dû au fait que ces questions naturelles sont toujours ouvertes mais aussi à cause des implications qu'auraient leurs résolutions. Je pense notamment à la théorie AdS/CFT à laquelle les métriques Poincaré-Einstein sont liées, mais plus simplement, à la compréhension des applications harmoniques dans un espace-temps. En tous cas, ils occuperont une part significative de mes recherches ces prochaines années.

Ce mémoire ne contient aucune preuve détaillée des différents résultats, j'invite donc le lecteur à ce reporter à l'article correspondant s'il souhaite les consulter. J'ai toutefois exposé au début de chaque chapitre des *baby cases* afin d'illustrer mon propos sans trop alourdir le manuscrit.

1.2 English version

This dissertation intends to present the various works I have done since my PhD and to give some perspectives on my future research.

During my PhD, I have in particular developed a pointwise analysis for blow-up phenomena for the constant mean curvature equation (CMC). Subsequently, I became interested in a larger set of equations : conformally invariant problems. If I had to define my work today in a few words, I would say that I'm doing blow-up analysis for conformally invariant problem. Conformal invariance plays a fundamental role in a large number of problems arising from physics (General Relativity, Conformal Field Theory, Turbulence, etc.), just like geometry (Conformal Geometry, Riemann Surfaces Theory, Yang-Mills Theory, etc.). As we will see in the chapter 3, this invariance by conformal transformation offers a particular structure to the equations giving birth to phenomena such as compactness by compensation. However, it also causes intrinsic analytical difficulties in the equations that govern these problems, such as their criticality and blow-up phenomena. The difficulty and the beauty of these equations lies in the fact that both the problems offered by their study but also their solutions are related to their very essence.

The most basic example of such a problem is the one of harmonic maps. Indeed they are the natural generalization of the problem of geodesics. It is considered as the fundamental example in calculus of the variations, of which I recall here the functional

$$E(u) = \int_{\Sigma} |\nabla u|_g^2 \, dv_g,$$

where u takes values into a fixed manifold N. If N is considered as a sub-manifold of $\mathbb{R}^m$, the Euler-Lagrange equation can be written as follows

$$\Delta u \perp T_u N.$$

Despite the simplicity of the formulation, as we will see in the course of the dissertation, the regularity and the study of the behavior of sequences of the solutions has been proved to be a very difficult problem. For regularity in dimension 2 (dimension of conformal invariance), we had to wait for Hélein's work in the early 90's and the first blow-up analysis has been done by Parker in the middle of the same decade. These fundamental advances can not be recalled here without mentioning the precursor work of Wente in 69 and Brezis-Coron in 82, for the CMC equation :

$$\Delta u = -2u_x \wedge u_y.$$

This equation is central in geometric analysis since it has its origin in the isoperimetric problem. The central questions for these problems are, after the one of existence, uniqueness and regularity of course, the question of the compactness of sequences of solutions and when this does not take place, the analysis of blow-up phenomena, from an energy point of view first and finally from a pointwise point of view. My research has been guided by this need to explore fine properties of fundamental objects such as harmonic maps, CMC surfaces and other variants, no less radical, in geometric analysis such as minimal surfaces, Willmore surfaces and the variety of problems that come within the rich and complex framework of conformal invariance in calculus variations.

For this memoir I have divided my work into 5 parts, but the reader will easily notice that all these works are strongly related.

The chapter 2 is dedicated to the Pohožaev identity. This choice is not meaningless, even if it corresponds to my oldest work. Indeed, we will find this identity, as well as its numerous variants, throughout the disertation as an essential argument in the analysis of conformally invariant problems. Moreover, we can already note its use by Uhlenbeck in her seminal work on the point-removability [**ScU**, **SU**]. I have also developed in this thesis a generalization of this idea which is based on the close link between the Pohožaev identity and the linearized solutions of the nonlinear problem, more particularly by reinterpreting them through the Noether's theorem. This idea is developed in section 5.1 and used in the following sections to fill the lack of Pohožaev identity (or equivalently of flux formula) for the Willmore equation. .
This chapter, therefore, begins with a brief introduction to the "classical" Pohožaev identity and the resulting obstruction for the Yamabe equation, whose I recall general form here

$$\Delta u + hu = u^{2^*-1},$$

where 2^* is the critical exponent for Sobolev injection of H^1. The criticality of the exponent makes it naturally subject to some concentration phenomena. The analysis of these phenomena has been a major challenge in the study of the space of constant scalar curvature metric, see [**BM**]. It also allows us to introduce Schoen's blow-up analysis strategy for this scalar equation, which will be a basis for vectorial equations. In sections 2.2 and 2.3, I present the results, obtained with Druet then Druet-Hebey, on the stability of the Pohožaev obstruction, a question initially raised by Brezis and Willem [**BW**]. We show that this obstruction persists with respect to $C^{0,\eta}$ perturbations of the linearity and $C^{1,\eta}$ of the domain. Moreover, this result is optimal since we also show the existence of solutions for perturbations in larger spaces. Finally I take advantage of this chapter to present, in section 2.4, a Pohožaev identity for 1/2-harmonic maps, obtained with Da Lio and Rivière. In the case of harmonic applications, the identity links the angular and radial parts of the gradient. For 1/2-harmonic maps, the formula involves a relation between the even and odd parts.

The chapter 3 presents the treatment of energy quantization phenomena for various problems. Here, to simplify, the goal is to show that the limit of the energies is equal to the sum of the energies of the limits. This naturally imposes a quantization of the energy when the limit solutions are classified. Beyond the resolution of open questions such as the quantization of biharmonic maps, I present a very robust strategy, developed with Rivière, which has found many applications to conformally invariant problems.
In the first four sections I introduce the class of conformally invariant problems, the precise concept of quantization (or energy identity) that we explore through concrete examples. Then in the sections 3.5 and 3.6, I outline our first result with Rivière, namely that any sequence of solutions of a conformally invariant problem in dimension 2 satisfies an energy identity. This very general result is born from a strategy which will subsequently be applied in many similar situations. The ingredients of the proof rest on three pillars : the conservation laws discovered by Rivière for the elliptic equations with antisymmetric potential [**Ri07**], the Lorentz spaces (with, as

keystone, a generalization of the Wente lemma on annuli whose conformal class degenerates), and finally the use of the Pohožaev identity.
In the last sections, I expose some quantization results for biharmonic maps, free boundary harmonics maps and finally their cousins namely the 1/2-harmonic maps.

The chapter 4 is dedicated to a study of solutions of conformally invariant problems when the conformal class of the underlying surface degenerates. After presenting the Deligne-Munford decomposition, which insures us that we can always find a conformal atlas consisting of a finite number of discs and collars, I present two quantization results in the absence of control of the conformal class. The first theorem is an improvement of the Gromov compactness result for the J-holomorphic curves, the second concerns harmonic maps satisfying a particular cohomological condition. Indeed there is no quantization in the general case for harmonic maps as already observed by Parker [**Pa**].
In a second part we are interested in the control of the conformal factor for the surfaces whose second fundamental form is bounded in L^2. This result generalizes the seminal works of Hélein and Müller-Šverák [**He**, **MuSv**]. They dealt with the case of the disc, we show that we can also control the conformal factor in the collars. For this we demonstrate, with Rivière, a uniform control of the Green function of the Laplacian in the L^2-weak space independently of the conformal class. We think that this result of great generality could find applications beyond the field of geometric analysis given the fundamental role played by the Green function of Riemann surfaces in many parts of mathematics.
In the next section, I present how, by relying on this last result, we proved the quantization for the Willmore surface sequences whose conformal class degenerates. As for the harmonic maps, without any additional hypothesis, there can be energy loss in the collar. However, we quantify this loss by a residue. The latter quantity seems crucial, in particular we show that it allows to distinguish Willmore surfaces which are conformally minimal from the one which are not, the Noether theorem plays also an important role in this last remark, since this residue has its origin in the conservation law given by invariance by rotation. I also present some results of compactness (when the conformal class is bounded), these being obtained using the residue given by the invariance by translation.
Finally I describe a work in preparation with Lira, whose goal is to achieve the first gluing of Willmore surfaces, with final objective to produce explicit examples of Willmore surfaces of bounded energy whose conformal class degenerates.

The chapter 5 presents a study of the behavior of sequences of CMC surfaces, and Willmore surfaces, in a Riemannian setting. More precisely we consider the situation of small surfaces which concentrate around a point and large surfaces in asymptotically Euclidean spaces. On the one hand, the presence of curvature makes these surfaces less rigid since, for instance, two round spheres can be glued to a CMC as proved by Zolotareava [**Zo**]. On the other hand, since the space is no longer homogeneous, the localization of these surfaces is constrained, as I had already shown in my PhD thesis for small CMC surfaces.
At the beginning of the chapter I explain how to overcome these difficulties via a fine analysis of the linearized operator. At first, it is necessary to obtain precise estimates despite the absence of maximum principle. Then always through the linearized operator and using the obtained estimates we can observe how the geometry constrains the location of these objects. The technique employed here generalizes in a way the

Pohožaev identity or, in a similar way, the flux formula.
The first result, obtained with Mondino, consists in showing that Willmore surfaces of energy smaller than 8π are necessarily concentrated around a critical point of the scalar curvature. We also extend this result to constrained area Willmore surfaces, which gives us an interesting corollary about the Hawking mass.
This link between Willmore's surfaces and general relativity is developed in the last section. I detail a program, in collaboration with Metzger, whose ultimate goal is to obtain a generalization of the Hawking mass. I show intermediate results on the uniqueness of the CMC and Willmore foliation obtained respectively by Huisken-Yau [**HuY**] and Lamm-Metzger-Schultze [**LMS1**].

The chapter 6 deals with the problem of the convexity of certain conformally invariant functionals. Indeed it is well known that as soon as the target manifold is positively curved, we lose the uniqueness of harmonic maps. However, as demonstrated by Colding and Minicozzi in [**CM08**], we can recover convexity and therefore uniqueness around critical points of small energies.
In a first section, I give a proof of this result which is based only on ε-regularity. This very robust proof has already found two applications to other conformally invariant problems that I expose in the following sections.
First of all, with Petrides, we have proved a local convexity for free boundary harmonic maps, which has allowed us to develop a replacement procedure in this setting and to obtain minimal free boundary surfaces realizing the width of a manifold with boundary.
Finally, with Lin, we have demonstrated this convexity of the energy in the case of biharmonic maps which take values into a sphere, this opens the door to the development of a remplacement procedure and then to some min-max theory in dimension 4.

This memoir only reflects the published part or in the process of being of my works. However, I would like to conclude the introduction with other axes of research that I am actively working on, but which to this day can not benefit of its own chapter.

— With Petrache, we have just pre-published a result on the rigidity of unidimensional uniformly distributed measure in any codimension. In fact it is a preparatory work to the case of surfaces which is a fairly open question. For the moment there is no clear argument in the sense of rigidity. This question of geometric measure theory is in fact very related to my works, since the problem is in some sense conformally invariant and that moreover its resolution passes by the choice of a suitable gauge of the Coulomb type.

— Another question I am actively working on is the study of harmonic applications which take value in a Lorentzian space. Indeed as I explain at the end of 3.3, this question is widely open. However, in recent months I have made significant advances, which could help us in understanding Willmore surfaces. Indeed, Bryant's correspondence insures us that we can associate to any Willmore surface a minimal surface in Sitter's space and consequently a harmonic map.

— As explained by Alexakis and Mazzeo in [**AM2**], the minimal surfaces in hyperbolic space can be interpreted as a special case of (Euclidean) Willmore surfaces with free boundary. However the ε-regularity at the conformal infinity does not seem completely understood. The techniques presented in this memoir for Willmore but also for free boundary problems are good candidates for solving this question. However, the problem seems very difficult, since even the question of the optimal regularity (that is to say C^1) of the boundary, already raised in [**deO**], remains open today.

— Previously in [**AM1**], Alexakis and Mazzeo motivated their study of minimal surfaces in hyperbolic space, highlighting the existence of a dictionary with the problem of Poincaré-Einstein metrics in dimension 4, which we can be briefly summarize in the following table

Dimension 2	Dimension 4
Minimal Surfaces in $\mathbb{H}^3$	Poincaré-Einstein Metrics
Renormalized Area	Renormalized Volume
$\int \lvert \mathring{A} \rvert^2$	$\int \lvert W \rvert^2$
Willmore surfaces in $\mathbb{R}^3_+$	Bach Metrics

As the recent paper of Chang and Ge [**ChGe**] shows, here again the ε-regularity at the conformal infinity is not completely understood. Although the problem is intrinsic, unlike those studied in this memoir, as it enjoys a conformal invariance through Weyl's energy, it should also be sensitive to the techniques I developed.

The topics discussed above are extremely active today, probably due to the fact that these natural issues are still open but also because of the implications of their resolutions. I am thinking in particular about the AdS/CFT theory to which the Poincaré-Einstein metrics are related, but more simply the understanding of harmonic maps which take value into a space-time. In any case, they will occupy a significant part of my research in the coming years.

This report does not contain any detailed proof of the various results, so I invite the reader to refer to the article if he wishes to consult them. However, I exposed at the beginning of each chapter of some baby cases to illustrate my point without weighing down the manuscript.

Chapitre 2

Stability of the Pohožaev Identity

This chapter is based on the following three articles : **(1)**, **(2)** and **(5)**.

2.1 The classical Pohožaev identity

Let us consider Ω a bounded domain of $\mathbb{R}^n$, with $n \geq 3$, and $1 < p \leq 2^* = \frac{2n}{n-2}$. We are interested in the existence of solution of the following problem :

$$\begin{cases} \Delta u + h(x)u = u^{p-1} & \text{on } \Omega \\ u > 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{P}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a given function. This equation appeared naturally in the 60′ independently in two domains. From one hand, in conformal geometry with the Yamabe problem which consists in looking for a constant scalar curvature metric in a given conformal class, from the other hand in theoretical PDE where one search solutions of the non-linear eigenvalue problem : $\Delta u = \lambda u^\alpha$.

When the exponent is subcritical, *i.e.* $1 < p < 2^*$, there is a lot of existing results, see **[Lions]**, for instance :

> If $1 < p < 2^*$ then (P) admits solutions if and only if $\Delta + h$ is coercive.

However, when $p = 2^*$ the condition is only necessary. From now on, we assume that $p = 2^*$ and that $\Delta + h$ is coercive. In the critical case, the leading order terms of the equation, that is to say $\Delta u = u^{2^*-1}$, become invariant under the conformal change $u \to \lambda^{\frac{2}{2^*-2}} u(\lambda\,.)$. This family of solutions generates a solution of the linearized operator which leads, by testing it against the equation, to the following constraint :

Theorem 1 (Pohožaev identity, [Poho]) *Let u a solution of (P) then*

$$\int_\Omega \left(h + \frac{1}{2}\langle x, \nabla h\rangle\right)^2 dx = -\frac{1}{2}\int_{\partial\Omega} \langle x, \nu\rangle (\partial_\nu u)^2\, d\sigma,$$

where ν is the exterior normal at $\partial\Omega$.

This result gives the following obstruction to the existence of solution of (P).

Corollary 2 (Pohožaev Obstruction) *If Ω is start-shaped and $h \in C^1(\Omega)$ satisfies*

$$h + \frac{1}{2}\langle x, \nabla h \rangle \geq 0, \tag{PO}$$

then (P) has no solution.

Clearly, $h = 0$ satisfies (PO) on $B(0,1)$, but the obstruction can not be extended to general domains, since we have an existence for $h = 0$ as soon as the Ω is not contractible, see **[BaCo]**.

2.2 Stability with respect to the linearity

In this section, we address the following natural issue :

Question 1 *Let Ω a start-shaped domain and $h \in C^1$ which satisfies (PO), if we perturb h, is the non-existence result preserved ?*

Of course we easily check that (PO) itself is not stable, since it is satisfied by 0 and not by any negative constant.

Before stating the main result of this section, let us remind the seminal work of Brezis and Nirenberg and the more recent work of Druet. It will give us some intuition on what we should expect.

A natural approach to find a solution of (P) consists in trying to minimize the following functional

$$E_h(u) = \frac{\int_\Omega |\nabla u|^2 + hu^2\,dx}{\left(\int_\Omega |u|^{2^*}\,dx\right)^{\frac{1}{2^*}}}.$$

We easily observe that

$$S_h = \inf_{u \in W_0^{1,2}(\Omega)} E_h(u) \leq K_n^{-2}, \tag{2.1}$$

where K_n is the best constant in the Sobolev injection : $W_0^{1,2} \hookrightarrow L^{2^*}$. As proposed by Aubin **[Aubin76]**, we can study more carefully the behavior of this functional against the following appropriate test functions :

$$u_\varepsilon(x) = \eta(x)\varepsilon^{\frac{2-n}{2}} U\left(\frac{x}{\varepsilon}\right),$$

where η is a cut-off function and

$$U(x) = \frac{1}{(1 + |x|)^{\frac{n-2}{2}}},$$

is, thanks to **[CGS]**, the unique, up to a constant, positive solution, of

$$\Delta u = u^{2^*-1} \text{ on } \mathbb{R}^n.$$

Then main idea, in order to prove existence, consists in using those test functions to prove that (2.1) is strict and to prove that the infimum is achieved. But the situation appears to be very different when either $n = 3$ or $n \geq 4$ as shown by the two following theorems.

Theorem 3 (Brezis-Nirenberg 83, [BN]) *If $n \geq 4$, then the following assertions are equivalent*

i) There exists $x \in \Omega$ such that $h(x) < 0$,

ii) $S_h < K_n^{-2}$,

iii) S_h is achieved by u_h a solution of (P).

In dimension 3, the situation is more complex, since we have :

Theorem 4 (Brezis-Nirenberg 83, [BN]) *If $n = 3, \Omega = B(0,1)$ there exists $\lambda^* < 0$ such that (P) admits no solutions for every $h = \lambda > \lambda^*$.*

An immediate consequence of theorem 3 is that there is no hope to get stability of the Pohožaev obstruction in a reasonable space for $n \geq 4$. While theorem 4 insures stability in a very special case : $\Omega = B(0,1)$ and h constant. We have to wait until 2002, to get a complete characterization of the minimizing solutions.

Theorem 5 (Druet 02, [Druet02]) *If $n = 3$, then the following assertions are equivalent*

i) There exists $x \in \Omega$ such that $g_h(x,x) > 0$,

ii) $S_h < K_n^{-2}$,

iii) S_h is achieved by u_h a solution of (P).

Here g_h denotes the regular part of G_h Green function of $\Delta + h$, i.e. $G_h(x,y) = \frac{c}{|x-y|} + g_h(x,y)$.

So in dimension 3, the previous theorem leaves *a priori* enough room to expect to get some stability. In order to be more precise in our statement let us define the following notion of stability.

Definition 1 *Let Ω be a star-shaped domain of $\mathbb{R}^3$ and let $(X, \|\,.\,\|_X)$ be some Banach space of functions on Ω (typically $X = C^{k,\eta}(\Omega)$, $X = L^\infty(\Omega)$ or $X = L^p(\Omega)$). Let $h_0 \in X \cap C^1(\Omega)$ be a function which satisfies (PO). We say that the Pohožaev obstruction is X-stable at (h_0, Ω) if the following property holds : there exists $\delta(h_0, \Omega, X) > 0$ such that for any function $h \in X$ with*

$$\|h - h_0\|_X \leq \delta(h_0, \Omega, X) \ ,$$

the equation (P) has no solution.

We say that the Pohožaev obstruction is X-stable if it is X-stable at (h_0, Ω) for all Ω star-shaped with respect to the origin and all $h_0 \in X \cap C^1(\Omega)$ satisfying (PO).

With Druet, we gave a complete answer to the question of the stability in the following theorems.

Theorem 6 (Druet-Laurain 10, (1)) *The Pohožaev obstruction is $C^{0,\eta}$-stable for any $\eta > 0$ in dimension 3.*

Note that a consequence of our theorem is the following : if Ω is a star-shaped domain in $\mathbb{R}^3$, there exists a constant $\hat{\lambda}(\Omega) > 0$ such that equation (P) has no solution with $h \equiv \lambda$ for $\lambda > -\hat{\lambda}(\Omega)$. This is in sharp contrast with the situation for non star-shaped domains, see **[BaCo]** for instance.

In [**BW**], Brezis and Willem studied this question in the case of the unit ball with radial functions. If we let

$$L_r^p(B) = \{u \in L^p(B) \, , \ u \text{ radial} \} \, ,$$

then it was proved that the Pohožaev obstruction is L_r^∞-stable [1] on the unit ball of $\mathbb{R}^3$ for all functions $h \in L_r^\infty(B) \cap C^1(B)$. In [**BW**], the question of the extension of the result to the non-radial case was explicitly asked. Our result provides an answer to this question. However, the situation is more delicate than expected in the non-radial case since, while the Pohožaev obstruction is $C^{0,\eta}$-stable for all $\eta > 0$, it is never L^∞-stable.

Theorem 7 (Druet-Laurain 10, (1)) *The Pohožaev obstruction is never L^∞-stable. In other words, given any $\varepsilon > 0$, any domain Ω in $\mathbb{R}^3$, star-shaped with respect to the origin and any function $h_0 \in C^1(\Omega)$ satisfying (PO), we can find some function $h_\varepsilon \in L^\infty(\Omega)$ such that*

$$\|h_\varepsilon - h\|_\infty \le \varepsilon$$

and some positive functions $u_\varepsilon \in C^2(\Omega)$ satisfying the equation

$$\begin{cases} \Delta u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^5 \ \textit{in} \ \Omega, \\ u_\varepsilon = 0 \ \textit{on} \ \partial\Omega, \\ u_\varepsilon > 0 \ \textit{in} \ \Omega. \end{cases}$$

Thus the L_r^∞-stability result obtained by Brezis and Willem is really specific to the radial case. In fact, it is not really due to the symmetry of the solutions but to one of its by-product in dimension 3, precisely that sequences of solutions of equation (P) which are radial are either compact or develop **only one concentration point**. In fact, with the PDE techniques (to be compared to the ODE techniques used in [**BW**]) we use in **(1)**, we can revisit the question of the stability of the Pohožaev obstruction in dimension 3 in the radial case. We improve the result of [**BW**] by proving that the Pohožaev obstruction is L_r^p-stable on the unit ball for all $p > 3$ but is never L_r^3-stable.

Theorem 8 (Druet-Laurain 10, (1)) *Let B be the unit ball of $\mathbb{R}^3$. Let h_0 be a C^1-radial function which satisfies (PO). Then for any $p > 3$, there exists $\delta(h_0, p) > 0$ such that if $h \in C^{0,\eta}(B)$ for some $\eta > 0$ with $\|h - h_0\|_{L^p(B)} \le \delta$, then there exists no solution of equation (P) in the unit ball.*

All these results give a complete picture of the stability of the Pohožaev obstruction in dimension 3 when the attention is restricted to non-negative solutions. The question remains widely open if one allows solutions to change sign, and is certainly more subtle due to the variety of sign-changing solutions of $\Delta u = u^5$ in $\mathbb{R}^3$, see [**RV1**] and [**RV2**] for instance.

The proof of theorem 6 and the one of the corresponding result in the radial situation, make use of standard blow-up analysis in dimension 3. Assuming the existence of a sequence of solutions u_ε associated to some h_ε converging to h_0, this sequence must blow-up, else it will converge to a solution of (P) which will contradict the fact that h_0 satisfies (PO). A careful analysis as the one initially developed by Druet and then Druet-Hebey-Robert, see [**Druet**] and [**DHR**], allows us to show that there is only finitely many concentration points and moreover none of them converges to the boundary. This last point is crucial and consists in one of the big improvement

1. One should restrict oneself to radial solutions of the equation in the definition of stability.

of our work with respect to the previous analysis. Then, relying on the previous analysis, we use an extension of Pohožaev's identity to a sum of Green's functions to reach a contradiction.
In order to prove that our result is optimal, we construct examples of functions h arbitrarily close in X to some given function which satisfies (PO) for which there is a solution of equation (P). It appears to be quite subtle because we need to be sharp. For instance, in order to prove Theorem 7, our functions h must be close to the given function in $L^\infty(\Omega)$ but not in $C^{0,\eta}(\Omega)$ for any $\eta > 0$.

2.3 Stability with respect to the domain

When we have started to give talks about the result developed in the previous section, the question of the stability with respect to the star-shaped condition naturally arose. More precisely :

Question 2 *Let Ω a start-shaped domain and $h \in C^1$ which satisfies (PO), if we perturb Ω is the non-existence result preserved ?*

Of course we can easily check that the condition to be start-shaped itself is not stable, as shown on the following picture :

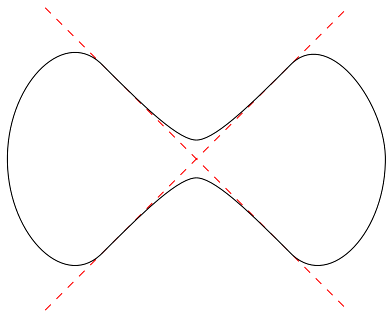

FIGURE 2.1 – Instability of the start-shaped condition

First of all let us extend our definition of stability to the domain.

Definition 2 *Let $\eta \in]0,1[$, Ω be a start-shaped smooth bounded domain of $\mathbb{R}^3$ and $h_0 \in C^1(\mathbb{R}^3)$ which satisfies (PO). We say that (Ω, h_0) is $C^{1,\eta}$-stable if there exists $\delta(h_0, \Omega, \eta) > 0$ such that for all $h \in C^{0,\eta}(\mathbb{R}^3)$ and $\Phi \in C^{1,\eta} \cap \mathrm{Diff}(\mathbb{R}^3)$ which satisfy*

$$\|h - h_0\|_{C^{0,\eta}} + \|\Phi - Id\|_{C^{1,\eta}} \leq \delta$$

then the following problem has no solution,

$$\begin{cases} \Delta u + hu = u^5 \text{ in } \widetilde{\Omega}, \\ u > 0 \text{ in } \widetilde{\Omega}, \\ u = 0 \text{ on } \partial\widetilde{\Omega}, \end{cases} \tag{2.2}$$

where $\tilde{\Omega} = \Phi(\Omega)$.

In collaboration with Druet and Hebey, we proved the following result.

Theorem 9 (Druet-Hebey-Laurain 13, (2)) *Let $\eta \in]0,1[$, Ω be a start-shaped smooth bounded domain of $\mathbb{R}^3$ and $h \in C^1(\mathbb{R}^3)$ which satisfies (PO) then (Ω, h) is always $C^{1,\eta}$-stable.*

The proof of this theorem is similar to the one of theorem 6. However a deeper attention has to be given to the boundary, especially to exclude concentration point that converge to it. In fact it appears in the proof that our previous argument fails as soon as the convergence is not in C^1.

In fact the hypothesis of this theorem are optimal in the sense that we can find a Lipchitz perturbation of the ball which admits positive solutions, as proved by the following theorem

Theorem 10 (Druet-Hebey-Laurain 13, (2)) *There exists Φ_ε a smooth diffeomorphism of $\mathbb{R}^3$ closed to identity in any $C^{0,\eta}$ such that the following problem admits a solution*

$$\begin{cases} \Delta u = u^5 \text{ in } \Omega_\varepsilon, \\ u > 0 \text{ in } \Omega_\varepsilon, \\ u = 0 \text{ on } \partial\Omega_\varepsilon, \end{cases} \tag{2.3}$$

where $\Omega_\varepsilon = \Phi_\varepsilon(B(0,1))$.

In fact, the technique works for any Ω which is axially symmetric and can probably be extended to any domain up to technical efforts. Our examples are built adding in a small "hook" to our ball, as follows.

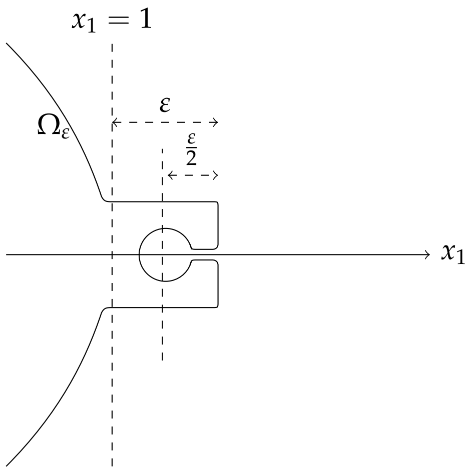

FIGURE 2.2 – Perturbation of the ball

The form was suggested to us by the previous work of Passasseo [**Pass**].

2.4 More about the Pohožaev identity

The Pohožaev identity gives rise to some more general obstruction in conformal geometry, in fact it is equivalent to the Kazdan-Warner identity in the problem of prescribed scalar curvature, see [**DR**].

An other important application of the Pohožaev idea was given by Schoen in dimension $n > 2$ with the following result.

Theorem 11 *Let (M^n, g) a manifold of dimension $n > 2$ with boundary Σ and X a vector field. Then*

$$\int_M X.(R_g)\, dv = -\frac{n}{n-2}\int_M \langle \mathring{Ric}_g, \mathcal{L}_X g\rangle\, dv + \frac{2n}{n-2}\int_\Sigma \mathring{Ric}(X,\nu)\, d\sigma, \tag{2.4}$$

where $\mathring{Ric}_g$ is the trace free part of the Ricci tensor, R_g is the scalar curvature, $\mathcal{L}_X$ is Lie derivative and ν is the outward unit normal vector field.

This identity had many consequences especially in the context of general relativity, see for instance **[BLF]** and reference inside. As we will develop in section 5.3, this identity has notably permitted to get a new definition of the mass which depends only of the curvature, see**[H]**. Let (M, g) be an asymptotically flat manifold and $X = r\partial_r$ the radial vector field in the chosen chart at infinity, then

$$m(g) = -\frac{-1}{(n-1)(n-2)\omega_{n-1}}\lim_{r\to+\infty}\int_{\partial B(0,r)}\left(\mathrm{Ric} - \frac{R_g}{2}g\right)(X,\nu)\, d\sigma_g.$$

We will make use of this identity in the context of Willmore surfaces into asymptotically flat manifold, as it is suggested in **[LMS]**. Indeed as soon as the mass is positive, the translation invariance of the space is broken and the Schoen-Pohožaev identity forces the Willmore sphere to be "globally centered".

It is also an important tool, when we study quantization phenomena, see chapter 3. In fact it is strongly linked to the conformal invariance as we show it on the following baby case.

Let us consider an harmonic map $u : \mathbb{D} \to M \subset \mathbb{R}^n$. Then it satisfies

$$\Delta u \perp T_u M. \tag{2.5}$$

By invariance of the Dirichlet energy, for every λ, $u_\lambda = u(\lambda\, .)$ is a family of solutions, then differentiating with respect to $\lambda = 1$, we obtain $v = x^i\partial_i u$ as a solution of the linearized operator, moreover it is trivially orthogonal to the Laplacian in our case. Hence we can test it again (2.5), which gives

$$\begin{aligned}
0 &= \int_{B(x_0,\rho)} \langle v, \Delta u\rangle\, dx = \int_{B(x_0,\rho)} \langle \nabla v, \nabla u\rangle\, dx \\
&= \int_{B(x_0,\rho)} |\nabla u|^2\, dx + \int_{B(x_0,\rho)} x^i\partial_i\left(\frac{|\nabla u|^2}{2}\right) dx - \frac{1}{\rho}\int_{\partial B(x_0,\rho)} |v|^2\, d\sigma \\
&= \frac{1}{\rho}\int_{\partial B(x_0,\rho)} r^2\frac{|\nabla u|^2}{2} - |v|^2\, d\sigma \\
&= \frac{\rho}{2}\int_{\partial B(x_0,\rho)} (\partial_r u)^2 - \left(\frac{\partial_\theta u}{r}\right)^2 d\sigma.
\end{aligned} \tag{2.6}$$

Hence the conformal invariance gives rise to a strong link between the radial derivatives and the angular derivatives. This link can be viewed has a "weak" conformal property. It will be a key point in the analysis of quantification phenomena, see chapter 3.

We have also generalized this notion to 1/2-harmonic maps, which are critical points of

$$E^{1/2}(u) = \int_{\mathbb{R}} |(-\Delta)^{1/4} u|^2 \, dx.$$

In fact this energy is conformally invariant, in the sense that it is invariant by the restriction to $\mathbb{R}$ of the action of the Möbius group on the Poincaré half-plane $\mathbb{H}$, where $\mathbb{R}$ is considered as its conformal infinity. With Da Lio and Rivière, we obtained the following results.

Theorem 12 (Da Lio-Laurain-Rivière 16, (5)) *Let $u \in W^{1,2}(\mathbb{R}, \mathbb{R}^m)$ be such that*

$$\left\langle \frac{du}{dx}, (-\Delta)^{1/2} u \right\rangle = 0 \;\; a.e \; in \; \mathbb{R}. \tag{2.7}$$

Assume that

$$\int_{\mathbb{R}} |u - u_0| dx < +\infty, \;\; \int_{\mathbb{R}} \left| \frac{du}{dx}(x) \right| \, dx < +\infty \tag{2.8}$$

Then the following identity holds for all $t \in \mathbb{R}$

$$\left| \int_{\mathbb{R}} \frac{x^2 - t^2}{(x^2 + t^2)^2} u(x) dx \right|^2 = \left| \int_{\mathbb{R}} \frac{2xt}{(x^2 + t^2)^2} u(x) dx \right|^2 . \tag{2.9}$$

The conditions (2.8) are automatically satisfied by the 1/2-harmonic maps which take value into a compact C^2 sub-manifold. By means of the stereographic projection we get an analogous formula in S^1.

Theorem 13 (Da Lio-Laurain-Rivière 16, (5)) *Let u be a $W^{1,2}$ map from S^1 into $\mathbb{R}^m$ be such that*

$$\frac{du}{d\theta} \cdot (-\Delta)^{1/2} u = 0 \qquad a.\, e. \; on \; S^1 \tag{2.10}$$

then the following identity holds

$$\left| \int_0^{2\pi} u(\theta) \, \cos\theta \, d\theta \right|^2 = \left| \int_0^{2\pi} u(\theta) \, \sin\theta \, d\theta \right|^2 . \tag{2.11}$$

In dimension 1 one might wonder what corresponds to the dichotomy in dimension 2 between **radial** and **angular** parts. Using the symmetry of (2.9), the next table is intended to illustrate the following correspondence of dichotomies respectively dimensions 1 and 2.

	2D		1D		
radial :	$\frac{\partial u}{\partial r}$	$\longleftrightarrow$	**symetric** part of u	:	$u^+(x) := \frac{u(x)+u(-x)}{2}$
angular :	$\frac{\partial u}{\partial \theta}$	$\longleftrightarrow$	**antisymetric** part of u	:	$u^-(x) := \frac{u(x)-u(-x)}{2}$

Indeed our **Pohožaev identity in dimension** 1 (2.11) can be rewritten as a **balancing law** between the **symmetric** part and the **antisymmetric** part of u.

$$\left| \int_0^{2\pi} u(\theta) \, \cos\theta \, d\theta \right|^2 = \left| \int_0^{2\pi} u(\theta) \, \sin\theta \, d\theta \right|^2 \tag{2.12}$$

This law is not invariant under the action of the Möbius group but the condition (2.10) is. Applying for instance a rotation by an arbitrary angle $\alpha \in \mathbb{R}$, the identity (2.11) implies

$$\begin{cases} |u_1| = |u_{-1}| \\ u_1 \cdot u_{-1} = 0 \end{cases} \tag{2.13}$$

where

$$\begin{cases} u_1 := \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \, \cos\theta \, d\theta \\ u_{-1} = \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \, \sin\theta \, d\theta. \end{cases}$$

The previous implies that there are "as many" Pohožaev identities as elements in this Möbius group minus the action of rotations, that is there are a $3 - 1 = 2$ dimensional family of identities exactly as in the 2-dimensional case where there are exactly as many **Pohožaev identities** (2.6) as choices of center $x_0 \in \mathbb{R}^2$ and radius $r > 0$ (which is again a $2 + 1 = 3$ dimensional space).

The proof of this identity relies on some integration by part against a fundamental solution of the heat equation. A similar idea has been preformed in **[CS]** to study the heat flow.

Those results are enlightened in the survey article **[DaL18]**. There also are some different approaches, due to Ros-Oton and Serra, see **[RS]**.

Coming back to the case of dimension 2, we can give a much richer version of (2.6), with the following theorem.

Theorem 14 (Da Lio-Laurain-Rivière 16, (5)) *Let $u \in W^{2,2}(\mathbb{R}^2, \mathbb{R}^m)$ such that*

$$\left\langle \frac{\partial u}{\partial x_i}, \Delta u \right\rangle = 0 \;\; a.e \text{ in } \mathbb{R}^2 \tag{2.14}$$

$i = 1, 2$. Then for all $x_0 \in \mathbb{R}^2$ and $t > 0$ the following identity holds

$$\int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} |x - x_0|^2 \left| \frac{\partial u}{\partial \nu} \right|^2 dx = \int_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} \left| \frac{\partial u}{\partial \theta} \right|^2 dx. \tag{2.15}$$

Chapitre 3

Quantization phenomena for conformally invariant problems

This chapter is based on the following four articles : **(3)**, **(4)**, **(5)** and **(6)**.

3.1 Introduction

In non-linear analysis, a very important question is the compactness of the set of critical points with a uniform bound on the energy. But of course this question turns out to be very difficult when the linearity is critical (with respect to the Sobolev embedding). In fact strong compactness is even false in general, that is why we often start to study weak-compactness, that is to say with respect to the energy. The main idea of the quantification is that, even if there is some loose of compactness, the level of energy which limits of solutions can reach are constrained. In order to observe such a phenomena we always need some additional properties, in this chapter we are going to explain how to use one of the most natural : the conformal invariance.

Conformal invariance is a property shared by many problems from geometry and physics. During the last decades it became a special subject in non-linear analysis, see **[Hel3]** and **[Ri1]** for instance. Indeed conformal invariant elliptic Lagrangians [1] share many common properties : from one hand their Euler-Lagrange equation are critical with respect to the space where their solutions are naturally defined, from the other hand those equations are invariant by dilation and consequently their solutions are subject to some *concentration* phenomena.

Some very natural questions such as the regularity of solutions or the compactness of sequences of solutions cannot be solved by classical tools from elliptic theory but we need to make a precise analysis of the interactions between the leading order derivatives and the non-linear terms.

In chapter 2, we have been interested in a Yamabe type equation in dimension $n \geq 3$,

$$\Delta u + hu = f u^{2*-1},$$

1. see section 3.3 for a precise definition.

which turns out to be conformally invariant in some sense and subject to some concentration phenomena. The analysis of the bubbling has led to some fundamental works, we refer here to the nice survey [**BM**].

In this chapter, we are going to focus on vectorial equations, in this case we face a new difficulty : the Euler-Lagrange equation becomes a strongly coupled elliptic system. Hence, contrary to the scalar case we have **no maximum principle**, which prevents from using tools developed for Yamabe-type equations. For instance, if we consider the very natural case of harmonic maps in dimension 2, we had to wait for the work of Hélein, see [**He**], in the begin of the 90's, to get the regularity in the general case. The book of Hélein which treats this question in the large, is a jewel in mathematical literature which had a lot of influence on my work. If we go through this book, we find the genesis of his idea. Those ideas and those that will come in this chapter owe a lot to the one developed by Uhlenbeck : [**SU**], [**ScU**], [**U**] (see also [**Do2**]). At first, Hélein considers the simpler situation where the target is symmetric. In this case, our problem is subject to some new kind of invariance. Inspired by the so called Noether's theorem (see page 29 of [**H2**] or [**KS**] for a nice introduction), Hélein derives some divergence free quantities and proved the regularity. In the general case, such conservation laws do not exist globally, but inspired by the Gauge theory of Uhlenbeck, he was able to get some *local* conservation laws. To conclude we can say that the lack of maximum principle can be overcome by the presence of some additional structure, which, as we will see, come from the conformal invariance.

Before going through the general case, I would like to present the analysis of the mean curvature equation which will permit me to underline the difficulties and introduce some tools.

3.2 Archetypal Example : The soap bubbles problem

An isoperimetrical variation of the Plateau's problem, the soap bubbles problem, consists in looking for a surface of minimal area which bounds a given curve and encloses a given volume. This problem can be translated in term of calculus of variations as follows : given a closed curve $\Gamma \subset \mathbb{R}^3$, we look for a map $u : \mathbb{D} \to \mathbb{R}^3$ which minimizes

$$A(u) = \int_{\mathbb{D}} |u_x \wedge u_y| \, dz$$

among the set of u such that $u(\partial\mathbb{D}) = \Gamma$ and

$$V(u) = \frac{1}{3}\int_{\mathbb{D}} \langle u, u_x \wedge u_y \rangle \, dz = 1.$$

Indeed $V(u)$ is the volume of the cone passing through 0 and bounded by $u(\mathbb{D})$. Minima [2] (and more generally critical points) satisfy

$$\begin{cases} \Delta u = -2\lambda u_x \wedge u_y \\ \langle u_x, u_y \rangle = 0, |u_x|^2 = |u_y|^2 \\ u(\partial\mathbb{D}) = \Gamma \end{cases} , \qquad (E_\lambda)$$

2. When they exist, if Γ is too big it is possible that no u satisfies $V(u) = 1$.

where λ is a Lagrange multiplier. This equation is of course critical in the following sense : if $u \in H^1(\mathbb{D})$, the natural space to minimize A, then $u_x \wedge u_y \in L^1$ and the standard elliptic theory doesn't apply. There is even a counter example to the regularity when the right hand side is only in L^1 :

$$u(z) = \ln\left(-\ln\left(\frac{|z|}{2}\right)\right) - \ln\left(-\ln\left(\frac{1}{2}\right)\right) \in H^1_0(\mathbb{D})$$

which satisfies

$$\Delta u = |\nabla u|^2.$$

But, as it was already said, we can take advantage of the particular form of the right hand side of (E_λ) which is a Jacobian and prove that this equation is subject to some compactness by compensation phenomena. This phenomena was initially discovered by Wente, proving the following fundamental lemma (see also **[BC1]**).

Lemma 15 (Wente 69, [We]) *There exists a universal constant $C > 0$ such that if $a, b \in W^{1,2}(\mathbb{D})$ and $u \in W^{1,1}_0(\mathbb{D})$ is a solution of the following equation*

$$\Delta u = a_x b_y - a_y b_x,$$

then

$$\|u\|_\infty + \|\nabla u\|_2 \leq C\|\nabla a\|_2 \|\nabla b\|_2.$$

Then this phenomena was intensively studied by many authors like Coifman, Lions, Meyer and Semmes **[CLMS]**, Tartar **[Tar]** (see also **[He]** et **[Ri1]** for some reviews). The problem of the optimal constant is also very interesting, with a strong geometric meaning, see **[Ge]** and **[Top]**.

Using notably this key lemma, Brezis and Coron proved that the soap bubbles problem admits, under reasonable assumptions, at least two solutions, see **[BC1]** or **[Str]**. However some natural questions are still open. For instance, in **[BC2]** Brezis and Coron have performed the first blow-up analysis for this equation, and proved some weak compactness. But, as a generalization of the plateau problem, we expect some strong compactness, that is to say

Conjecture 1 *Let (u_n) a (H^1-bounded) sequence of $H^1(\mathbb{D})$ of solution of (E_λ), then (u_n) is pre-compact in $C^2(\mathbb{D})$, up to the action action of the conformal group.*

However this question is still wide open today. With Druet we get some partial result we did not publish, because of some unnatural conditions on the boundary.

As already noticed, contrary to scalar equations, we have no maximum principle in this setting. For instance, we do not even know if the symmetry of the boundary propagates in the interior, and the following conjecture, proposed by Brezis and Coron in 84 **[BC1]** is still wide open :

Conjecture 2 (Spherical caps conjecture) *If Γ is a circle, then (E_λ) admits only two solution* [3].

3. The small and large part of the sphere which bound Γ.

There is nevertheless some partial result but with a quite strong assumption, which permits to apply the Alexandrov maximum principle, see [**RR**].

In some sense, those two longstanding conjectures underline quite well the level of difficulty we reach in this setting.

3.3 Conformally invariant problems

Here we will define more precisely what is a conformally invariant problem. We will limit ourself to dimension 2, since the conformal group is bigger and then the equations are simpler. But all the concepts can easily generalized to higher dimension.

Definition 3 *Let Σ a Riemann surface*[4] *and $N \subset \mathbb{R}^N$ a sub-manifold. We say that a functional L is conformally invariant if*

$$L(u) = \int_\Sigma l(u, \nabla u)\, dv_\Sigma$$

where $l : \mathbb{R}^m \times \mathbb{R}^2 \otimes \mathbb{R}^m \to \mathbb{R}$ and $u \in W^{1,2}(\Sigma, N)$, and such that

(i) for all $\phi \in \mathrm{Conf}(\Sigma)$ with deg$(\phi) = 1$, we have $L(u \circ \phi) = L(u)$,

(ii) there exists $C > 0$ such that

$$\frac{|p|^2}{C} \leq l(\,.\,, p) \leq C|p|^2,$$

where $\mathrm{Conf}(\Sigma)$ is the the space of conformal map from Σ to itself.

Here are some examples.

(a) If $E(u) = \int_\Sigma |\nabla u|^2\, dv_\Sigma$, the critical points are the so-called **harmonic maps**, they satisfy

$$\Delta u = A(u)(\nabla u, \nabla u),$$

where A is the second fundamental form of N.

(b) If $\Sigma = \mathbb{D}$, $N = \mathbb{R}^3$ and let $H : \mathbb{R}^3 \to \mathbb{R}$, we set

$$E(u) = \int_{\mathbb{D}} |\nabla u|^2\, dz + \frac{1}{3} \int_{\mathbb{D}} Q(u).u_x \wedge u_y\, dz,$$

where $Q : \mathbb{R}^3 \to \mathbb{R}^3$ with $\mathrm{div}(Q) = H$. The critical points are **surfaces with prescribed mean curvature** equal to H (H-surfaces), they satisfy

$$\Delta u = -2H(u) u_x \wedge u_y,$$

when u is conformal.

(c) Let (N, ω, J) a Kähler manifold and $E(u) = \int_\Sigma u^*(\omega)$. The critical points are the J**-holomorphic curves**, they satisfy

$$\frac{\partial u}{\partial x} = J(u) \frac{\partial u}{\partial y},$$

where $z = x + iy$ are some holomorphic coordinates.

4. Since most problems are local, we can restrict to $\mathbb{D}$.

All those equations are critical, as (E_λ), in the sense they are of the form

$$\Delta u \sim |\nabla u|^2.$$

The regularity of harmonic maps was shown by Hélein **[He]**, for H-surfaces by Wente **[We]** and Hildebrandt **[Hi]** in the general case, and for the J-holomorphic curves by Sikorav **[Si]**. Hildebrandt made the following conjecture

Conjecture 3 (Hildebrandt conjecture [Hil]) *: Every critical point of a conformally invariant functional is smooth.*

In fact, thanks to the result of Giusti-Miranda and Morrey, see Theorem 9.8 of **[GM]**, it suffices to prove that they are continuous. However, the conjecture had to wait for the work of Rivière in 2006. In the following we briefly give some details about Rivière's work, in order to introduce some key material.

Initially it was noted by Grüter **[Gru]** that all conformally invariant functionals share a common algebraic structure. Then Rivière extended the result to the Euler-Lagrange equation.

Theorem 16 (Gruter 84 -Rivière 07) *Let $u \in W^{1,2}(\mathbb{D}, N)$ a critical point of conformally invariant functional, then there exists $\Omega \in L^2(\Lambda^1\mathbb{D} \otimes so(m))$ such that*

$$-\Delta u = \Omega.\nabla u\,. \tag{*}$$

For instance the prescribed mean curvature equation

$$\Delta u = -2H(u)u_x \wedge u_y$$

can be written in the form (*) with

$$\Omega = H(u)\begin{pmatrix} 0 & -\nabla^\perp u^3 & \nabla^\perp u^2 \\ * & 0 & -\nabla^\perp u^1 \\ * & * & 0 \end{pmatrix}.$$

It is very important to note that contrary to the case when H is constant, the right hand side cannot be put in divergence form without assuming H very regular. In particular the compactness by compensation theory does not apply directly to this equation.

In fact Rivière showed that not only every critical point of a conformally invariant is regular but even any solution of (*).

Theorem 17 (Rivière 07, [Ri07]) *Let $u \in W^{1,2}(\mathbb{D}, \mathbb{R}^N)$ which satisfies (*), then for every $p > 2$ we have $u \in W^{1,p}(\mathbb{D}, \mathbb{R}^N)$.*

The main idea of the proof consists in rewriting the equation, in order to make appear some Jacobian structure and to apply Wente's lemma. This relies on the existence of a non-linear Hodge decomposition in the spirit of the work of Uhlenbeck on Yang-Mills equation.

Theorem 18 (Theorem I.3 [Ri07]) *There exists $\varepsilon_0 > 0$ such that if $\|\Omega\|_2 \leq \varepsilon_0$ then there exists $A \in L^\infty \cap W^{1,2}(\mathbb{D}, GL_n)$ and $B \in W^{1,2}(\mathbb{D}, M_n)$ such that*

$$\nabla A - A\Omega = \nabla^\perp B$$

In this situation, (*) becomes

$$\begin{cases} \operatorname{div}(A\nabla u) = \nabla^\perp B \nabla u \\ \operatorname{curl}(A\nabla u) = \nabla^\perp A \nabla u \end{cases} .$$

We have to note that the invertibility and the L^∞-estimate of A is crucial, since Wente inequality gives *a priori* only some estimate on $A\nabla u$. This is the main reason why we need the antisymmetry of Ω, since $so(n)$ is seen as the Lie algebra of $SO(n)$ and this is the compactness of this space which permits to get the L^∞-estimate on A. Moreover the result becomes false if Ω is in the Lie algebra of a non compact group as we can see it in section 6 of Zhu [**ZhuL**], with $\Omega \in so(1,1)$.

In fact, Zhu was interested in the case of harmonic maps into de Sitter when he did this remark. This raises the following question

Question 3 *Are harmonic maps into a Lorentzian manifold regular ? And more importantly, do they satisfy some ε-regularity ?*

If the manifold is stationary, we can split the space and time part and reduce the question to the Riemannian case, as it was done in [**ZhuL**]. Zhu treats also the case of de-Sitter which possesses natural conservation laws. But the general case seems totally open since (for the moment) there is no existence of a Coulomb frame in $SO(1,n)$ because of the non-compactness of this group.

To conclude we can say that the conformal invariance is responsible of the regularity, forcing Ω to be antisymmetric. But this invariance is also responsible of an even more trivial phenomena : The equation are conformally invariant in the sense that : if $u : \mathbb{D} \to N$ is a solution $u_\lambda = u(\lambda\, .)$ is also a solution.

In particular they are subject to some concentration phenomena and then some loss of compactness. The study of these phenomena is the goal of the next section.

3.4 Weak Compactness : Energy Identity

Let us start by giving a heuristic of the phenomena of loss of compactness or *bubbling* that come from the seminal work of Sacks and Uhlenbeck on minimal immersions (see [**SU**]).

Let $u_n : \mathbb{D} \to N$ be a sequence of solution of conformally invariant problem. One can think about some harmonic maps or CMC disc to fix ideas.

Let us assume that this sequence is non-compact, then $\|\nabla u_n\|_\infty \to +\infty$. Otherwise by standard elliptic theory, u_n must converge in C^2_{loc}.

For sake of simplicity we can also assume that $\nabla u_n(0) = \|\nabla u_n\|_\infty$. Hence we define $\tilde{u}_n : B\left(0, \frac{1}{\lambda_n}\right) \to N$, by

$$\tilde{u}_n = u_n(\lambda_n\,.),$$

where $\lambda_n = \|u_n\|_\infty^{-1}$. Then $\|\nabla \tilde{u}_n\|_\infty = 1$ and, by elliptic theory, $\tilde{u}_n$ converges, up to extraction, to $\omega \not\equiv 0$ in $C^2_{loc}(\mathbb{R}^2)$. Moreover ω satisfies the same equation but this time **on the whole plane**, such a solution is called a **bubble** [5].
Finally we assume that the concentration occurs only at 0, then $u_n \to u_\infty$ in $C^2_{loc}(\mathbb{D} \setminus \{0\})$, we even know, thanks to some point removability argument, that if $\|\nabla u_n\|_2$ is bounded then $u_\infty \in C^2_{loc}(\mathbb{D})$.

To summarize, we control very well the behavior of u_n far from 0 but also at the scale λ_n. A natural question consists in understanding what happens in the intermediary region, the **neck region** [6]. In particular what happens in term of energy ? Is the whole lost energy, when u_n goes to u_∞, is absorbed by the bubble or is there some energy loss in the neck region, that is to say do we have

$$\lim_{R\to+\infty} \lim_{n\to+\infty} \|\nabla u_n\|_{L^2(B(0,1/R)\setminus B(0,R\lambda_n))} = 0 \ ?$$

In this case it is refereed as an **Energy identity** , since

$$\lim_{n\to+\infty} \|\nabla u_n\|_2^2 = \|\nabla u_\infty\|_2^2 + \|\nabla \omega\|_2^2.$$

Of course we just have to consider the case of one bubble, in general there are several which concentrate, even at the same point, and we also neglected the possibility to concentrate at the boundary. For the sake of simplicity, this last phenomena will not be considered in this section, but we will give some results about concentration at the boundary in section 3.6. Here is a rigorous definition of an energy identity.

Definition 4 (Energy identity) *We say that a sequence of solutions $u^n \in W^{1,2}(\mathbb{D}, N)$ satisfies an* **energy identity** *if there exists*

- — $u^\infty \in W^{1,2}(\mathbb{D}, N)$ *a solution of the equation,*
- — $a_1^n, \dots, a_k^n \in \mathbb{D}$ *finitely many points of concentration,*
- — $\lambda_1^n, \dots, \lambda_k^n \in \mathbb{R}^+$,
- — $\omega_1, \dots, \omega_k$ *bubbles*

such that

$$u^n \to u^\infty \text{ in } W^{1,2}_{loc}(\mathbb{D} \setminus \{a_i^\infty, \dots, a_k^\infty\}),$$

$$\tilde{u}_i^n = u^n(a_i^n + \lambda_i^n z) \to \omega_i \text{ in } W^{1,2}_{loc}(\mathbb{R}^2 \setminus S_i)$$

where S_i is finite [7]*, and*

$$\left\| \nabla \left(u^n - u^\infty - \sum_{i=1}^k \omega_i \left(\frac{\cdot - a_i^n}{\lambda_i^n} \right) \right) \right\|_2 \to 0.$$

If we consider our examples of conformally invariant problems, in dimension 2, each one already possesses a proof of an energy identity. For the harmonic maps, it is due

5. The terminology comes from the fact that for the CMC equation the bubble are round spheres.
6. It is the region where the bubble is glued to the limit solution.
7. It consists of the trace of the other concentration points on the rescaled domain

to Parker **[Pa]** and the best assumption known before our work is N of class C^3 and the main tool considered here was the Hopf differential. For the problem of H-surfaces, a first proof was given by Brezis and Coron **[BC2]**, when H is constant. In the general case the best known result is the one of Caldiroli and Musina **[CM]** assuming $H \in C^1$ with some decay at infinity, the main tool here was the Wente inequality. Finally for J-holomorphic curves, the first proof is due to Gromov **[Gr85]** and the best assumption known are N of class C^2, the proof relies on an isoperimetric inequality.

Our initial goal, with Rivière, was to give a proof of these different results using only the conformal invariance. Indeed, in addition to shedding new light on conformally invariant problems, a better understanding of the mechanism of compactness by compensation could allow us to reach more natural hypotheses for the mentioned examples. For instance, there is no reason to restrict to some H function which decreases at infinity in the prescribed mean curvature problem. Moreover, such a proof could be generalized to higher dimension, which is hopeless with tools such as the Hopf differential, coming from complex analysis (there are specific to the dimension 2).

As Rivière proved the regularity for solutions of (*), and not only of critical points of conformally invariant functional, in the next section we study energy identity for sequences of solutions of (*).

3.5 Energy identity for elliptic systems with antisymmetric potential

First, it should be noted that energy identities are usually characteristic of non-linear phenomena, but here we are interested in a linear systems, which gives some originality to the result. Then, it is clear that one can not directly transpose the techniques resulting from the regularity for the following reason : to prove regularity, we always assume that we are under a certain level of energy to ensure the existence of conservation laws, as in Theorem 18. But when we have a concentration phenomenon, we are sure to be above this level of energy [8]. Nevertheless, the result about regularity can be slightly improved to an ε-regularity result. This concept, coming back to Schoen-Uhlenbeck work, say roughly speaking that under a certain level of energy everything is under control. More precisely, we prove the following

Theorem 19 *There exists $\varepsilon_0 > 0$ and $C > 0$ such that if $\Omega \in L^2(\Lambda^1\mathbb{D}, so(n))$ (resp. $L^2(\Lambda^1\mathbb{R}^2, so(n))$) satisfies*

$$\|\Omega\|_2^2 \leq \varepsilon_0,$$

then

1. *(ε-regularity) If $u \in W^{1,2}(\mathbb{D}, \mathbb{R}^n)$ satisfies*

$$-\Delta u = \Omega.\nabla u \text{ on } \mathbb{D} \tag{3.1}$$

8. at least the one of the ε-regularity.

then we have the following estimate

$$\sup_{B_{\frac{1}{2}}(0)} |\nabla u| \leq C\|\nabla u\|_2.$$

2. *(Energy gap) The only function $u \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^n)$ which satisfy*

$$-\Delta u = \Omega.\nabla u \text{ on } \mathbb{R}^2 \tag{3.2}$$

are the constant functions.

In order to prove our result, we will have to carry out a fine analysis in the neck regions, for that we will notably use a refinement of the L^p-spaces, Lorentz spaces. We refer to Chapter 3 of **[He]** or Section 1.4 of **[Gra]** for some precise definition.

The only thing to remember about Lorentz-spaces, is that we have a refinement of L^2 given by $L^{2,1} \subset L^2 \subset L^{2,\infty}$, with the same scaling property, such that $\frac{1}{|z|} \in L^{2,\infty}$ and $L^{2,1}$ and $L^{2,\infty}$ are duals[9]. Those spaces are particularly well adapted to scale invariant problems. In fact the invariance by dilation and the fact that $\frac{1}{|z|} \in L^{2,\infty}$ give us naturally some no-neck property for the $L^{2,\infty}$-norm. Let us a give a very brief sketch of this argument in order to understand how general it is.

Let us assume that we have a sequence $u_n : \mathbb{D} \to N$ such that

$$\lim_{n\to+\infty} \|\nabla u_n\|_{L^2(\mathbb{D}\setminus B(0,\lambda_n))} \leq \varepsilon_0, \tag{3.3}$$

where $\varepsilon_0 > 0$ is given by the ε-regularity. Then we claim that, for any $2\lambda_n < r_n < 1/2$, we have

$$\lim_{n\to+\infty} \|\nabla u_n\|_{L^{2,\infty}(B(0,2r_n)\setminus B(0,r_n/2))} = 0.$$

Indeed, otherwise if we can catch some energy in a dyadic region, we could extract a sequence which converges to a non-trivial solution on $\mathbb{R}^2$, using the point removability, which would contradicted the energy gap thanks to (3.3).
Therefore, applying the ε-regularity in a dyadic region, we deduce that $|\nabla u_n| = o\left(\frac{1}{z}\right)$ and we get the no-neck property for the $L^{2,\infty}$-norm.

Therefore, our problem reduces to get some $L^{2,1}$-bound and a first step in this direction is given by the following improvement of the Wente inequality.

Lemma 20 (Improved Wente) *Let $a, b \in W^{1,2}(\mathbb{D})$ and $\phi \in W_0^{1,1}(\mathbb{D})$ a solution of*

$$\Delta\phi = a_x b_y - a_y b_x,$$

Then

$$\|\nabla\phi\|_{2,1} \leq C\|\nabla a\|_2\|\nabla b\|_2.$$

9. Not exactly, we have $(L^{2,1})' = L^{2,\infty}$ but only $L^{2,1} \subset (L^{2,\infty})'$

This inequality contains the original one, since contrary to $W^{1,2}$, $W^{1,(2,1)}$ is embedded into L^∞ (C^0 actually). This inequality can be seen as a consequence of the work of Coifman, Lions, Meyer et Semmes **[CLMS]** who showed that $a_x b_y - a_y b_x \in \mathcal{H}^1$, the Hardy space (see section 3.4 of **[He]** for more detail).

The main difficulty of our work consisted in generalizing this kind of inequality to some annuli with some bound that does not depends on the conformal class of the annulus, since the neck regions consist precisely in degenerating annuli. During our analysis we discovered that the radial part and the angular part of the gradient behaves differently. This can be interpreted as the fact that the Green function of the Laplace operator, which is given by $(x,y) \mapsto \ln(\sqrt{x^2+y^2})$, is radial. Here is the key lemma.

Lemma 21 *Let $0 < \varepsilon < \frac{1}{2}$ and $f : B_1 \setminus B_\varepsilon \rightarrow \mathbb{R}$ an harmonic function, and*

$$c = \frac{1}{2\pi} \int_{\partial B_1} \partial_\nu f \, d\sigma. \tag{3.4}$$

Then, for all $\lambda > 1$, there exists positive a constant $C(\lambda)$ independent of ε and f such that

$$\|\nabla(f - c\ln(r))\|_{L^{2,1}(B_1 \setminus B_{\lambda\varepsilon})} \leq C(\lambda)\|\nabla f\|_2.$$

Thanks to this fundamental estimate we were able to give an annuli version of the improved Wente inequality and to quantified the energy of the angular derivative. But we were also able to describe very precisely the possible loss of energy for the radial derivatives, see Theorem 2 & Section 3.3 of (3) for precise statement. Here is condensed version

Theorem 22 (Laurain-Rivière 13, (3)) *The angular derivative of a bounded sequence of solutions of a* **linear system** *with* **antisymmetric** *potential satisfies an energy identity. There exists a sequence for which the radial derivatives does not satisfy an energy identity, in this case the loss of energy is measured by a residue.*

The existence of a non-zero residue comes from the fact that the conservation laws can not be global, that is to say that at the level of the neck one cannot extend them inside the disk otherwise the residue would be trivial. This residue phenomena will also appears when we will treat the case of degenerating conformal class, since we will have a thin-thick decomposition with disc and annuli, see chapter 4.

However, in the case of existence of global conservation laws this result is enough to get the full quantification of the energy. Indeed if we have a conservation law on the whole disc, the residue automatically vanishes and the radial derivatives are also quantized. This idea was used by Bernard and Rivière for Willmore surfaces. In fact Rivière has discovered, by computation, a conservative form of the Willmore equation in **[Riv08]**, then this has been re-proved by Bernard **[B]**, taking advantage of the conformal invariance of the Willmore functional and the Noether theorem. Hence Bernard and Rivière using notably Lemma 21 and the existence of those global conservation laws were able to prove the quantization of the Willmore energy when the conformal class stays bounded, see **[BR]**. We will discuss more deeply the quantization phenomena for Willmore surfaces in Chapter 4, especially when the conformal class degenerates.

This purely analytical result found many applications to more geometric problems including a total quantification of energy, which is the aim on the next section.

3.6 Applications

Conformally invariant problem in dimension 2

Thanks to Theorem 16, if u is a critical point of a conformally invariant functional then u satisfies
$$-\Delta u = \Omega.\nabla u$$
with Ω antisymmetric. But the conformal invariance also forces the right hand side to be orthogonal to ∇u (see Theorem I.2 **[Ri1]**) :

$$\Omega.\nabla u \text{ is orthogonal to } u_x \text{ et } u_y.$$

Consequently, u satisfies a Pohožaev identity, see chapter 2 for more details.

$$\int_{\partial B_r} \left|\frac{\partial u}{\partial r}\right|^2 dz = \int_{\partial B_r} \left|\frac{\partial u}{r\partial \theta}\right|^2 dz$$

In particular the control of the energy of the angular derivatives obtained in Theorem 22 implies directly a control of the energy of the whole gradient.

Theorem 23 (Laurain-Rivière 13, (4)) *In dimension* 2*, every bounded sequence of critical points of a conformally invariant functional satisfies an energy identity.*

In addition to its great generality, this result allows us to improve a certain number of existing results. Indeed the proof only requires $\Omega \in L^2$, which is the minimal hypothesis in order the equations make sense. In particular, we have been able to give optimal versions of the existing results for our examples :

(1) Harmonic equations : $\Omega = A(\nabla u, .)$, we only need $A \in C^0$, i.e $N \in C^2$.

(2) Prescribed mean curvature problem : $\Omega \sim H(u)\nabla u$, we only need $H \in C^0$.

(3) J-holomorphic curves : we only needs $J \in C^0$, i.e $N \in C^1$.

The last result is even global, see section 4.2. Beyond these improvements, the proof relying only on the conformal invariance can easily be generalized to problems of higher order and of higher dimension as we will see it in the following.

Biharmonic maps in dimension 4

Let $(N, g) \subset \mathbb{R}^N$ be a closed C^3 Riemannian submanifold of $\mathbb{R}^N$. The *extrinsic bi-energy* of a $W^{2,2}$ map $u = (u^1, ..., u^n) : B_1 \to N$ from the 4-dimensional unit ball $B_1 = B_1(0) \subset \mathbb{R}^4$ is defined by

$$E(u) = \frac{1}{4}\int_{B_1} |\Delta u|^2 dx\,, \tag{3.5}$$

where Δ is the Laplacian operator on $\mathbb{R}^4$. A weakly *extrinsic bi-harmonic map* u from B_1 into $N \hookrightarrow \mathbb{R}^N$ is a map in $W^{2,2}(B_1, \mathbb{R}^N)$, which is a critical point of $E(u)$ and takes values almost everywhere in N. Similarly, a weakly *intrinsic bi-harmonic map* $u \in W^{2,2}(B_1, N)$ is a critical point of the *intrinsic bi-energy*

$$I(u) = \frac{1}{4}\int_{B_1} |(\tau(u)|^2 dx = \frac{1}{4}\int_{B_1} |\Delta u|^2 - |A(u)(\nabla u, \nabla u)|^2 dx\,, \qquad (3.6)$$

where $\tau(u) = (\Delta u)^T$ is the tangential component of Δu, which is commonly called the *tension field*, and $A(u)$ is the second fundamental form of the embedding of N in $\mathbf{R}^N$. Both the extrinsic and intrinsic bi-energies are scaling invariant in 4 dimensions. Now let $\Pi : N_\delta \to N$ be the nearest point projection map, which is well defined and C^3 for $\delta > 0$ small enough. Here $N_\delta = \{y \in \mathbb{R}^N \mid d(y, N) \leq \delta\}$. For $y \in N$, let

$$P(y) \equiv \nabla\Pi(y) : \mathbb{R}^N \to T_y N$$

be the orthogonal projection, and

$$P^\perp(y) \equiv \mathrm{Id} - \nabla\Pi(y) : \mathbb{R}^N \to (T_y N)^\perp\,.$$

In the following, we will write P (resp. $P^\perp$) instead of $P(y)$ (resp. $P^\perp(y)$) and we will identify these linear transformations with their matrix representations in $\mathcal{M}_n$. We also note that these projections are in $W^{2,2}(B_1, \mathcal{M}_n)$ as soon as u is in $W^{2,2}(B_1, N)$. Finally, note that the second fundamental form $A(\cdot)(\cdot,\cdot)$ of $N \subset \mathbb{R}^N$ is defined by

$$A(y)(Y, Z) = D_Y P^\perp(y)(Z), \quad \forall\, y \in N \quad \text{and} \quad Y, Z \in T_y N.$$

We know that $u \in W^{2,2}(B_1, N)$ is an *extrinsic bi-harmonic map* if and only if (see e.g. **[Wa1]**)

$$\Delta^2 u \perp T_u N \text{ almost everywhere,}$$

which can be rewritten as follows :

$$\begin{aligned} \Delta^2 u = &- \Delta(\nabla P^\perp \nabla u) - \mathrm{div}\,(\nabla P^\perp \Delta u) + 2\nabla P^\perp \nabla(\nabla P^\perp \nabla u) \\ &+ 2\nabla P^\perp \nabla P^\perp \Delta u - (\nabla P P^\perp - P^\perp \nabla P)\nabla\Delta u. \end{aligned} \qquad (3.7)$$

For *intrinsic bi-harmonic map*, we need to add some tangent terms (see **[Wa2]** and (4) for detail) and the equation is

$$\begin{aligned} \Delta^2 u = &- \Delta(\nabla P^\perp \nabla u) - \mathrm{div}\,(\nabla P^\perp \Delta u) + 2\nabla P^\perp \nabla(\nabla P^\perp \nabla u) + 2\nabla P^\perp \nabla P^\perp \Delta u \\ &- (\nabla P P^\perp - P^\perp \nabla P)\nabla\Delta u + P\left(\langle \nabla P^\perp \nabla u, \nabla_u(\nabla_u P^\perp)\nabla u \nabla u\rangle\right) \\ &+ \langle \nabla P^\perp \nabla u, \nabla_u P^\perp \nabla u \nabla P\rangle - \mathrm{div}\,\langle \nabla P^\perp \nabla u, \nabla_u P^\perp \nabla u P\rangle \\ &+ \langle \nabla P^\perp \nabla u, \nabla P^\perp \nabla P\rangle - \mathrm{div}\,\langle \nabla P^\perp \nabla u, \nabla P^\perp P\rangle\,. \end{aligned} \qquad (3.8)$$

Here $\nabla_u P^\perp = \nabla_y P^\perp(y)|_{y=u}$. Alternatively, when $N = \mathbf{S}^n$, the intrinsic bi-harmonic map u satisfies the equation (see e.g. Lamm-Rivière **[LammR]**)

$$\Delta^2 u = \Delta(V \cdot \nabla u) + \mathrm{div}\,(w \nabla u) + W \cdot \nabla u\,, \qquad (3.9)$$

where

$$\begin{cases} V^{ij} = u^i \nabla u^j - u^j \nabla u^i \\ w^{ij} = \operatorname{div}\left(V^{ij}\right) \\ W^{ij} = \nabla w^{ij} + 2\left[\Delta u^i \nabla u^j - \Delta u^j \nabla u^i + |\nabla u|^2 (u^i \nabla u^j - u^j \nabla u^i)\right] . \end{cases} \tag{3.10}$$

It is worth noting that intrinsically the intrinsic bi-harmonic map equation (3.8) can be written as

$$\Delta_g^2 u - R^N(\nabla_g u, \tau(u)) \nabla_g u = 0 \,,$$

where R^N is the curvature tensor of N. This was first derived by Jiang in **[Jiang86]**.

Harmonic maps are clearly critical points (absolute minimizers) of the intrinsic bi-energy $I(u)$ since they satisfy $\tau(u) = 0$. In this sense, the intrinsic bi-harmonic map is a more geometrically natural generalization of the harmonic map, although the extrinsic bi-harmonic map is more natural from the analytical point of view. It shall be noted that the extrinsic bi-energy $E(u)$, in contrast to $I(u)$, depends on the embedding of the target manifold N in $\mathbf{R}^N$. Despite the fact that it is more geometrically natural to study the intrinsic bi-harmonic map, it is less studied and considerably more difficult analytically than the extrinsic bi-harmonic map due to the lack of *coercivity* for the intrinsic bi-energy $I(u)$ (because the two terms Δu and $A(u)(\nabla u, \nabla u)$ in the intrinsic bi-energy $I(u)$ can compensate but cannot dominate each other in an analytic way). The existence and non-existence of nontrivial (or *proper*, *i.e.*, non-harmonic) *intrinsic* bi-harmonic maps can be found in Jiang **[Jiang86]**, Mou **[Mou00]**, Caddeo-Montaldo-Oniciuc **[CMO01, CMO02]**, Oniciuc **[Oni02]** and Baird-Kamissoko **[BK03]**. For the regularity of weakly intrinsic bi-harmonic maps on 4 dimensional Euclidean domains, see e.g. Ku **[Ku08]** for spherical targets and Moser **[Mos06]**, Wang **[Wa1]** for general targets. The regularity of weakly *extrinsic* bi-harmonic maps has been studied by Chang-Wang-Yang **[CWY99]**, Strzelecki **[Str03]** and Wang **[Wa1, Wa2]** (see also Lamm-Rivière **[LammR]**).

Remark 1 *There is a notion of bi-harmonic sub-manifolds (intrinsic bi-harmonic isometric immersion $\mathcal{M} \to N$), which generalizes the notion of minimal sub-manifolds (harmonic isometric immersion $\mathcal{M} \to N$) since every harmonic map is an intrinsic bi-harmonic map. It is conjectured that any bi-harmonic sub-manifold in sphere has parallel mean curvature vector (commonly referred to as the generalized Chen's conjecture), see Ou* **[Ou16]** *for a survey.*

Remark 2 *The divergence terms $-div\,\langle \nabla P^\perp \nabla u, \nabla_u P^\perp \nabla u P\rangle$ and $-div\,\langle \nabla P^\perp \nabla u, \nabla P^\perp P\rangle$ were missing in the intrinsic bi-harmonic map equation (3.8) in Wang* **[Wa2]** *and all other related literatures. The analytical estimates needed for regularity still hold for these missing terms and thus the results in* **[Wa2]** *etc. are still valid.*

Generalizing methods developed in dimension 2, and developing the initial work of Lamm and Rivière **[LammR]**, we were able to prove the following theorem.

Theorem 24 (Laurain-Rivière 13, (4)) *Any bounded sequence of (intrinsic or extrinsic) biharmonic maps satisfy an energy identity.*

This result was partially known in the intrinsic case, see **[HM]**. Our method being robust to perturbation, we also obtain a quantification result for the biharmonic flow.

Let us describe briefly what is the biharmonic flow and its basic properties before stating our result.

The negative L^2-gradient flow of the extrinsic bi-energy $E(u)$ is called a (weakly) extrinsic bi-harmonic map heat flow. It is given by $u \in W^{1,2}([0,T), L^2(B_1,N)) \cap L^2([0,T); W^{2,2}(B_1,N))$ which satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} + \Delta^2 u = -\Delta(\nabla P^\perp \nabla u) - \operatorname{div}(\nabla P^\perp \Delta u) + 2\nabla P^\perp \nabla(\nabla P^\perp \nabla u) \\ \qquad\qquad + 2\nabla P^\perp \nabla P^\perp \Delta u - (\nabla P P^\perp - P^\perp \nabla P)\nabla \Delta u \quad \text{on } B_1 \times [0,T)\,; \\ \qquad u = u_0 \quad \text{on } B_1 \times \{0\}\,; \\ \qquad u = \chi(x)\,, \quad \partial_\nu u = \xi(x) \quad \text{on } \partial B_1 \times [0,T)\,. \end{cases} \tag{3.11}$$

where $u_0 \in W^{2,2}(B_1,N)$ and χ and ξ are adapted data. There are several results of existence for extrinsic bi-harmonic map heat flow, see for instance Lamm **[Lamm04]** for small initial data and Gastel **[Gas06]** and Wang **[Wa4]** for solutions with many finitely singular time and any initial data. Moreover, the solutions to (3.11) satisfy the following energy inequality

$$2\int_0^T \int_{B_1} \left|\frac{\partial u}{\partial t}\right|^2 dx\,dt + \int_{B_1 \times \{T\}} |\Delta u|^2\,dx \le \int_{B_1} |\Delta u_0|^2\,dx\,. \tag{3.12}$$

Similarly, a weakly intrinsic bi-harmonic map heat flow is the negative L^2-gradient flow of the intrinsic bi-energy $I(u)$ given by $u \in W^{1,2}([0,T), L^2(B_1,N)) \cap L^2([0,T); W^{2,2}(B_1,N))$ which satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} + \Delta^2 u = -\Delta(\nabla P^\perp \nabla u) - \operatorname{div}(\nabla P^\perp \Delta u) + 2\nabla P^\perp \nabla(\nabla P^\perp \nabla u) + 2\nabla P^\perp \nabla P^\perp \Delta u \\ \qquad - (\nabla P P^\perp - P^\perp \nabla P)\nabla \Delta u + P\left(\langle \nabla P^\perp \nabla u, \nabla_u(\nabla_u P^\perp)\nabla u \nabla u\rangle\right) \\ \qquad + \langle \nabla P^\perp \nabla u, \nabla_u P^\perp \nabla u \nabla P\rangle - \operatorname{div}\langle \nabla P^\perp \nabla u, \nabla_u P^\perp \nabla u P\rangle \\ \qquad + \langle \nabla P^\perp \nabla u, \nabla P^\perp \nabla P\rangle - \operatorname{div}\langle \nabla P^\perp \nabla u, \nabla P^\perp P\rangle \quad \text{on } B_1 \times [0,T)\,; \\ u = u_0 \quad \text{on } B_1 \times \{0\}\,; \\ u = \chi(x)\,, \quad \partial_\nu u = \xi(x) \quad \text{on } \partial B_1 \times [0,T)\,. \end{cases} \tag{3.13}$$

where $u_0 \in W^{2,2}(B_1,N)$ and χ and ξ are adapted data. The solutions to (3.13) satisfy the following energy inequality

$$2\int_0^T \int_{B_1} \left|\frac{\partial u}{\partial t}\right|^2 dxdt + \int_{B_1 \times \{T\}} |\tau(u)|^2\,dx \le \int_{B_1} |\tau(u_0)|^2\,dx\,. \tag{3.14}$$

Intrinsically, the intrinsic bi-harmonic map heat flow equation (3.13) can be written as

$$\frac{\partial u}{\partial t} + \Delta^2 u = R^N(\nabla u, \tau(u))\nabla u\,. \tag{3.15}$$

Contrary to the extrinsic bi-harmonic map heat flow, there is no bi-energy monotonicity for the intrinsic bi-harmonic map heat flow (cf. (3.12) and (3.14) for $T > 0$), hence in general only short time existence of a smooth solution is known for the intrinsic bi-harmonic map heat flow, see e.g. Lamm **[Lamm01]**, Mantegazza-Martinazzi

[MM12].

Improving the methods used in the previous theorem, we were able to prove the weak compactness of such flows.

Corollary 25 (Laurain-Rivière 13,(4)) *Let N be a compact submanifold of class C^3 in $\mathbb{R}^k$, $u_0 \in W^{2,1}(B_1, N)$ and $u \in W^{2,2}([0, +\infty[\times B_1, N)$ a global solution of (3.11) or (3.13). Then there exists a sequence of reals $t_n \to +\infty$, a biharmonic maps $u_\infty \in W^{2,1}(B_1, N)$, $l \in \mathbb{N}$, $\omega^1, \dots, \omega^l$ some biharmonic maps from $\mathbb{R}^4$ to N and $a_n^1, \dots, a_n^l$ a family of converging sequences of B_1, such that*

$$u(t_n, .) \longrightarrow u_\infty \text{ on } W^{2,p}_{loc}(B_1 \setminus \{a^1_\infty, \dots, a^l_\infty\}) \text{ for all } p \geq 1$$

and

$$\begin{aligned} &\left\| \nabla^2 \left(u(t_n, .) - u_\infty - \sum_{i=1}^{l} \omega_n^i \right) \right\|_{L^2_{loc}(B_1)} + \\ &\left\| \nabla \left(u(t_n, .) - u_\infty - \sum_{i=1}^{l} \omega_n^i \right) \right\|_{L^4_{loc}(B_1)} \to 0, \end{aligned} \tag{3.16}$$

where $\omega_n^i = \omega^i(a_n^i + \lambda_n^i\, .)$.

Application to free boundary harmonic maps and 1/2-harmonic maps

Another very active area of analysis where those methods apply is the one of non-local elliptic PDEs. We present here the case of 1/2-harmonic maps, see **[Sch]** for more general operators. The case of the 1/2-harmonic applications is particularly interesting since they are connected to the free boundary minimal surfaces and to the extremal metric problem for the first eigenvalue of the Dirichlet-to-Neumann operator, a subject also in expansion (see **[FS]** and **[Pe]**). Let us introduce the more common problem of free boundary harmonic maps and their links with 1/2-harmonic maps.

Let Σ be a surface with boundary and N a submanifold with boundary of $\mathbb{R}^N$. The critical points of the Dirichlet energy among all maps which map the boundary of the surface to the boundary of the submanifold, *i.e.*

$$\{u \in W^{1,2} \cap L^\infty(\Sigma, N) \mid u(\partial\Sigma) \subset \partial N\}$$

are called free boundary harmonic maps, there are harmonic on the interior of Σ and satisfies at the boundary Σ

$$\partial_\nu u \perp T_u \partial N,$$

where ν is the outward normal of Σ on $\partial\Sigma$.

In the special case $\Sigma = \mathbb{D}$, we also speak about 1/2-harmonic maps from S^1 into ∂N, since

$$\Delta^{\frac{1}{2}} u \perp T_u \partial N,$$

where $\widehat{\Delta^{\frac{1}{2}} u} = |\xi| \hat{u}$ and $\hat{u}$ is the Fourier transform of u. In fact we easily check that if $u : S^1 \to N$, then denoting its harmonic extension by $\tilde{u}$, we get $\partial_\nu \tilde{u} = \Delta^{\frac{1}{2}} u$. This extension theory of non-local operator was developed in **[CaSi]**. The disk case is also very important for its links with minimal surfaces. Indeed, considering the Hopf differential, a free boundary harmonic maps is automatically conformal, hence its image is minimal outside branch points (see **[DP]** for recent development).

Da Lio and Rivière recently developed a totally intrinsic point of view on 1/2-harmonic maps, generalizing the conservation laws approach of the local case, see **[DaLR]**. They proved the regularity and the ε-regularity and even some quantization phenomena in the special case of maps taking values into the sphere **[DaL]**.

In collaboration with Petrides, we generalized those results to free boundary harmonic maps defined on arbitrary surfaces. Our argument consists in extending the maps through the free boundary up to loss the harmonicity but preserving the antisymmetric structure as in (*).

Theorem 26 (Laurain-Petrides 15, (6)) *Let (Σ, g) a Riemann surface with boundary and $u_m \in W^{1,2}((\Sigma, g), \mathbb{B}^{n+1})$ a bounded sequence of free boundary harmonic maps. Then, there exists a free boundary harmonic maps $u_\infty : \Sigma \rightarrow \mathbb{B}^{n+1}$ and*

- *$\omega^1, \cdots, \omega^l$ a family $1/2$-harmonic maps from $\mathbb{R} \rightarrow \mathbb{S}^n$,*
- *$a^1_m, \cdots, a^l_m$ a family of points in $\partial\Sigma$,*
- *$\lambda^1_m, \cdots, \lambda^l_m$ some sequences of positive numbers ,such that,*

$$u_m \rightarrow u_\infty \text{ in } C^\infty_{loc}(M \setminus \{a^1_\infty, \cdots, a^l_\infty\}) ,$$

and

$$\int_{\partial\Sigma} R_m.\partial_\nu R_m \rightarrow 0 ,$$

with

$$R_m = u_m - u_\infty - \sum_{i=1}^{l} \omega^i \left(\frac{\cdot - a^i_m}{\lambda^i_m} \right) .$$

This result have been extended by Jost, Liu and Zhu **[JLZ]** to harmonic maps into general manifolds, see also proposition A.1 of **(9)** for an alternative proof. In the sphere case we take advantage of the symmetry to use an inversion to extend the maps, in the general we can still extend it locally through local symmetries, as already done by Scheven, see **[Scheven]**.

Independently we also treat the general case for 1/2-harmonic maps in collaboration with Da Lio and Rivière. Here the method relies also on a fine analysis in the neck regions and the discovering of a Pohožaev identity, see the end of Section 2.4.

Theorem 27 (Da Lio-Laurain-Rivière 17, (5)) *Let $u_n \in H^{1/2}(\mathbb{R}, N)$ be a sequence of $1/2$-harmonic maps such that $\|u_k\|_{\dot{H}^{1/2}} \leq C$. Then,*

1. *There exists $u_\infty \in H^{1/2}(\mathbb{R}, N)$ an $1/2$-harmonic maps and some sequences of points $\{a_1, \dots, a_\ell\}$, such that, up to extraction,*

$$u_n \rightarrow u_\infty \quad \text{in } W^{1/2,p}_{loc}(\mathbb{R} \setminus \{a_1, \dots, a_\ell\}), \ \ p \geq 2$$

2. *There exists $\omega^i_\infty \in \dot{H}^{1/2}(\mathbb{R})$ some $1/2$-harmonic maps, such that, up to extraction,*

$$\left\| (-\Delta)^{1/4} \left(u_n - u_\infty - \sum_i \tilde{\omega}^i_n \right) \right\|_{L^2_{loc}(\mathbb{R})} \rightarrow 0 .$$

Chapitre 4

Control of surfaces with degenerating conformal class

This chapter is based on the following three articles : **(3)**, **(7)** and **(8)**.

In the previous chapter we have treated the problem of quantization from a local point of view, that is to say on disks. This is enough if the conformal class of the surface stays bounded, but if we want to treat the problem in full generality, we can't cover the surface by disks with radius bounded from below, and we have to consider a thin-thick decomposition. The goal of the next section is to give a brief description of what happens when the conformal class of a surface degenerates.

4.1 Deligne-Mumford decomposition

Here we recall the Deligne-Mumford's description of the loss of compactness of the conformal class for a sequence of Riemann surfaces with fixed topology, see **[Hum]** for details.

Let Σ be a closed smooth surface of genus g. We can endow Σ with a metric h_0, then thanks to the uniformization theorem, see **[Do]**, there exists in the conformal class [1] of h_0 a metric h of constant curvature, equal to 1 if $g = 0$, to 0 if $g = 1$ and to -1 otherwise. We assume that Σ is endowed with such a metric. The sphere case is very peculiar, since the conformal group is not compact but this case is not of great interest here since there is only one conformal class. In the hyperbolic case the metric is unique and in the torus it is also true up to normalize the area.

Let (Σ, c_k) be a sequence of closed Riemann surfaces of fixed genus g and let $c_k = [h_k]$ be the conformal class. If $g = 0$ then the conformal class is fixed. If $g = 1$ then, we know that (Σ, c_k) is conformally equivalent to

$$\mathbb{R}^2 / \left(\frac{1}{\sqrt{\Im(v_k)}} \mathbb{Z} \times \frac{v_k}{\sqrt{\Im(v_k)}} \mathbb{Z} \right),$$

1. *i.e.* the set of metric on Σ which can be written $e^{2u} h_0$ where u is a smooth function

where v_k lies in the fundamental domain $\{z \in \mathbb{C} \text{ s.t. } |\Re(z)| \leq 1 \text{ and } |z| \geq 1\}$ of $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$, and we say that c_k degenerates if $|v_k| \to +\infty$. If $g > 1$, let h_k the hyperbolic metric associated with c_k, then (Σ, c_k) degenerates if there exits a closed geodesic whose length goes to zero. In that case, up to a subsequence, there exists

1. an integer $N \in \{1, \dots, 3g-3\}$,
2. a sequence $\mathcal{L}_k = \{\Gamma_k^i \,;\, i = 1 \dots N\}$ of finitely many pairwise disjoint simple closed geodesics of (Σ, h_k) with length converging to zero,
3. a closed Riemann surfaces $(\overline{\Sigma}, \overline{c})$,
4. a complete hyperbolic surface $(\widetilde{\Sigma}, \widetilde{h})$ with $2N$ cups $\{(q_1^i, q_2^i) \,;\, i = 1 \dots N\}$ such that $\widetilde{\Sigma}$ was obtained topologically after removing the geodesics of $\mathcal{L}_k$ to Σ and after closing each component of the boundary of $\Sigma \setminus \mathcal{L}_k$ by adding a puncture q_l^i at each of these component. Moreover $\overline{\Sigma}$ is topologically equal to $\widetilde{\Sigma}$ and the complex structure defined by $\widetilde{h}$ on $\widetilde{\Sigma} \setminus \{q_l^i\}$ extends uniquely to $\overline{c}$. We can also equip $\overline{\Sigma}$ with a metric $\overline{h}$ with constant curvature, but not necessarily hyperbolic since the genus of $\overline{\Sigma}$ can be lower than the one of Σ.

The surface $(\widetilde{\Sigma}, \widetilde{h})$ is called the nodal surface of the converging sequence and $(\overline{\Sigma}, \overline{c})$ is its renormalization. These objects are related, in the sense that, there exists a diffeomorphism $\psi_k : \widetilde{\Sigma} \setminus \{q_l^i\} \to \Sigma \setminus \mathcal{L}_k$ such that $\widetilde{h}_k = \psi_k^* h_k$ converge in C^∞_{loc} topology to $\widetilde{h}$.

Morally, we can find an atlas which consists of finitely many disks that cover the *thick* part, that it is to say a region where the injectivity radius is bounded from below, and a long cylinder covering the *thin* part, the region where the injectivity radius is small. In the following picture, there is one collar and two punctured tori.

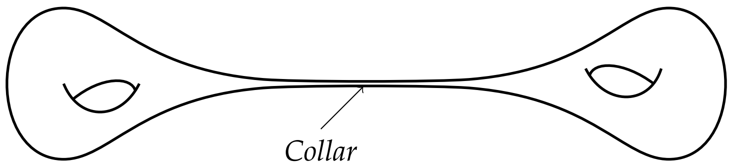

FIGURE 4.1 – Genus 2 surface degenerating with formation of a *collar*.

The collar region are conformally equivalent to hyperbolic cylinders.

4.2 Quantization of pseudo-holomorphic curves and harmonic maps on degenerating Riemann surfaces

We consider (N, J) to be a smooth almost-complex manifold and we look at pseudo-holomorphic curves between (Σ, h) and (N, J), in other words we consider applications $u \in W^{1,2}(\Sigma, N)$ satisfying

$$\frac{\partial u}{\partial x} = J(u) \frac{\partial u}{\partial y} \quad , \tag{4.1}$$

where $z = x + iy$ are some local conformal coordinates on Σ. These objects are fundamental in symplectic geometry, see **[McDS]**. In the study of the *moduli space* of pseudo-holomorphic curves in an almost complex manifold, the compactification question comes naturally. In other words it is of first importance to understand and to describe how sequences of pseudo-holomorphic curves with possibly degenerating conformal class behave at the limit.

As a by-product of the analysis develop in the previous chapter, in (2) we get a new proof of the so-called Gromov's compactness theorem [**Gr85**], see also [**Hum**]. Up to our knowledge, we improve the assumption on the regularity of the target manifold up to the optimal one.

Theorem 28 (Laurain-Rivière 14, (3)) *Let (N, J) be a C^1 compact almost complex manifold, Σ a closed surface and (c_n) a sequence of complex structures on Σ. Assume that $u_n : (\Sigma, c_n) \to (N, J)$ is a sequence of pseudo-holomorphic curves of bounded area with respect to an arbitrary metric on N. Then u_n converge weakly to some cusp curve*[2] $\overline{u} : \overline{\Sigma} \to (N, J)$ *and there exists finitely many bubbles, holomorphic maps $(\Omega^i)_{i=1\cdots l}$ from S^2 into (N, J), such that, modulo extraction of a subsequence*

$$\lim_{n\to+\infty} E(u_n) = E(\overline{u}) + \sum_{i=1}^{l} E(\omega^i) \quad . \tag{4.2}$$

In fact, the bound on the energy is not necessary if the target manifold is symplectic, i.e if there is a closed 2-form ω on N compatible with J. Indeed, in that case (see chapter 2 of [**McDS**] for instance) all $u : \Sigma \to N(J, \omega)$, regular enough, satisfy

$$A(u) = \int_\Sigma dvol_{u^*g} \geq \int_\Sigma u^*\omega$$

where $g = \omega(., J.)$, with equality if and only if u is pseudo-holomorphic. Hence, for symplectic manifolds, pseudo-holomorphic curves are area minimizing in their homology class. In particular, they are minimal surfaces, *i.e.* conformal and harmonic.

As it was explained in Chapter 3, our method, in comparison with previous ones such as the one given in [**Zhu**], has the advantage to require less regularity on the target manifold N. In fact, following the approach of [**Pa**], in order to establish the angular energy quantization, Zhu proved an estimate of the following second derivative

$$\frac{d^2}{d\theta^2} \int_{S^1\times\{t\}} |u_\theta|^2 \, d\theta$$

Such an estimate requires for the metric of N to be at least C^2. In the alternative proof we have provided, we only require the almost complex structure and the compatible metric to be C^1.

But *a priori* our result requires the conformal structure to be bounded. In fact, in the particular case of first order operator, it works even if the conformal class degenerates, thanks to the following lemma.

2. we refer to chapter 5 of [**Hum**] for precise definitions

Lemma 29 *Let $n \in \mathbb{N}^*$, $(a_i)_{1\leq i\leq n}$ and $(b_i)_{1\leq i\leq n}$ be two families of maps in $W^{1,2}(B_1)$, $0<\varepsilon<\frac{1}{4}$ and $\phi \in W^{1,2}(B_1\setminus B_\varepsilon)$ which satisfies*

$$\nabla\phi = \sum_{i=1}^{n} a_i\, \nabla^{\perp} b_i. \tag{4.3}$$

Then, for $0<\lambda<1$, there exists a positive constant $C(\lambda)$ independent of ϕ, a_i and b_i such that

$$\|\nabla\phi\|_{L^{2,1}(B_\lambda\setminus B_{\lambda^{-1}\varepsilon})} \leq C(\lambda)\left(\sum_{i=1}^{n}\|\nabla a_i\|_2\,\|\nabla b_i\|_2 + \|\nabla\phi\|_2\right).$$

Hence, for first order elliptic systems, we can replace Lemma 21 by this one and get the $L^{2,1}$ estimate on the whole gradient in each neck or collar region.

Quantization for harmonic maps on a degenerating surface, a cohomological condition

The aim of this section is to shed some new light on the quantization for harmonic maps on a degenerating surface, which was fully described by Zhu in **[Zhu]**. In this case some loss of energy is possible in the collar region and it is measured by a residue. However, when the target is a sphere, we were able to make a link between the energy quantization for harmonic maps and a cohomological condition.

Theorem 30 (Laurain-Rivière 12, (3)) *Let (Σ, c_n) be a sequence of closed Riemann surfaces. Let u_n be a sequence of harmonic maps from (Σ, h_n) into the unit sphere S^{N-1} of the euclidean space $\mathbb{R}^N$. Assume that*

$$\limsup_{n\to+\infty} E(u_n) < +\infty$$

and assume that the following closed forms

$$\forall i,j = 1\cdots m \qquad \star\,(u_n^i\, du_n^j - u_n^j\, du_n^i)$$

are all exact. Then the energy quantization holds : modulo extraction of a subsequence, on each component of the limiting thick part, u_n converges strongly, away from the punctures, to some limiting harmonic map u and there exists finitely many bubbles, holomorphic maps $(\omega^i)_{i=1\cdots l}$ from S^2 into S^{N-1}, forming possibly both on the thick and the thin parts, such that, modulo extraction of a subsequence

$$\lim_{n\to+\infty} E(u_n) = E(u) + \sum_{i=1}^{l} E(\omega^i) \quad . \tag{4.4}$$

More generally we can raise the following question.

Question 4 *Consider a sequence of harmonic maps from a degenerating surface to a general target manifolds. Is there a simple cohomological condition similar as the one in Theorem 30 ensuring the quantization of the energy in collar region ?*

4.3 Control of the conformal factor of surfaces with L^2-bounded second fundamental form

In this section we give a uniform estimate for the gradient of the Green function on a closed Riemann surface, independent of its conformal class, and we derive compactness results for immersions with L^2-bounded second fundamental form and for Riemannian surfaces of uniformly bounded Gaussian curvature entropy.

In the following we assume that Σ is a surface of genus $g \geq 1$ with a metric of constant curvature and we associate to h its Laplace-Beltrami operator Δ_h. Then there exists, up to normalization, a unique nonnegative Green function G_h associated to Δ_h.
With Rivière we obtained an estimates on G_h independently of the conformal class defined by the metric h. This is a very classical subject in the theory of Riemann surfaces strongly related with the behaviour of the spectrum of the Laplace operator, see [**Buser**].

Let (Σ, h_k) be a sequence of hyperbolic surfaces whose conformal classes degenerate, that is to say that some geodesics are pinching. Let us assume that there is only one degenerating geodesic γ_k and let denote Σ_∞ its nodal limit. Then Ji, see [**Ji**], proved that if γ_k does not separate Σ_k then G_k is uniformly bounded on every compact subset of $(\Sigma_\infty \times \Sigma_\infty) \setminus \Delta$, where Δ is the diagonal, else $\lim\limits_{k\rightarrow+\infty} |G_k(x,y)| = +\infty$ for any $(x,y) \in \Sigma_\infty \times \Sigma_\infty$.

Here we see that we have a very different behaviour with respect to the manner the conformal class degenerates. Is a similar behaviour possible for derivatives ?

Formally we can write

$$G_k(x,y) = \sum_{i\geq 1} \frac{\varphi_i^k(x)\varphi_i^k(y)}{\lambda_i^k},$$

where λ_i^k and φ_i^k are respectively the i-th (non vanishing) eigenvalue and the ith (non constant) eigenfunction of Δ_{h_k}, repeating indices according to multiplicity. Of course if the nodal limit is disconnected then the first eigenvalue goes to zero while the first eigenfunction goes to a locally constant function on each connected component. Looking at derivatives instead, one can expect a better behavior of the Green function, even in the collar region. However, the gradient of the Green function has a simple pole on the diagonal, hence it is not in L^2. The main result of (8) establishes that it is however true for a slightly weaker norm, which has the same scaling than $L^{2,\infty}$.

Theorem 31 (Laurain-Rivière 18, (8)) *Let Σ be a closed surface. Then there exits C a positive constant and an integer N depending only on the genus of Σ such that for any metric h on Σ with constant curvature equal to 1, 0 or -1 and with normalized volume*[3]*, and any Green function associated to h, there exists a finite atlas of N conformal charts (U_i, ψ_i), such that for any $y \in \Sigma$ we get*

$$\sup_{t>0} t^2 \left| \left\{ x \in V_i \mid |d_x G_h^i(x,y)| \geq t \right\} \right| \leq C, \tag{4.5}$$

3. 4π in the sphere case and 1 in the torus case

where $V_i = \psi_i(U_i)$ *and* $G_h^i(\,.\,,y) = (\psi_i)*(G_h(\,.\,,y))$.

We can remark that on a fixed Riemann surface (Σ, h), the Green function is always bounded for this weak L^2-norm. We can deduce it from the standard pointwise estimate, see **[Aubin]**,

$$|d_x(G_h^i(x,y))| \leq \frac{C_h}{d_h(x,y)},$$

but of course this estimate depends on the metric that we take on Σ.

This result is clearly false in L^2. In fact the atlas is very explicit, since, for instance considering the hyperbolic case, the surface divides in thick and thin part. On the thick part we can consider any disk with radius smaller than the injectivity radius. And in the thin part, using collar lemma, chart are given by degenerating annuli.

This result is optimal in the sense that we also prove that the weak L^2-norm of the Green function computed with respect to the intrinsic metric is not bounded when the singularity occurs on a collapsing region, see **(8)** for details .

Regarding the proof of the theorem, once we have ruled out the trivial case of the sphere, we treat the case of a degenerating torus and of an hyperbolic surface quite differently. For the torus, our proof relies on an estimate of the coefficient of the Fourier decomposition on a long thin cylinder using the periodicity condition, and in the hyperbolic case, it relies on the coarea formula and the decomposition of the surface into thin and thick parts.

We give an application of Theorem 31 in differential geometry both from extrinsic and intrinsic point of view. Indeed our initial goal was to understand the behaviour of the conformal factor for Willmore surfaces with bounded energy and more generally for surface with L^2-bounded second fundamental form. But we believe that our result should have application to many other fields due to the central place of the Green function.

First, we prove that the gradient of the conformal factor of an immersion with L^2-bounded second fundamental form is uniformly bounded in $L^{2,\infty}$, up to choosing a convenient atlas given by Theorem 31.

Theorem 32 (Laurain-Rivière 18, (8)) *Let* (Σ, c_k) *be a sequence of closed Riemann surfaces of fixed genus at least one. Let* h_k *denote the metric with constant curvature (and volume equal to one in the torus case) in* c_k *and* Φ_k *a sequence of weak conformal immersions of* Σ *into* $\mathbb{R}^N$*, i.e.*

$$\Phi_k^* \xi = e^{2u_k} h_k,$$

where $u_k \in L^\infty(\Sigma)$ *and* ξ *is the standard metric in* $\mathbb{R}^N$*. Then there exists a finite conformal atlas* (U_i, ψ_i) *independent of* k *and a positive constant* C *depending only on the genus of* Σ*, such that*

$$\|dv_k^i\|_{L^{2,\infty}(V_i)} \leq C W(\Phi_k),$$

where v_k^i is the conformal factor of $\Phi^k \circ \psi_i^{-1}$ in $V_i = \psi_i(U_i)$, i.e. $v_k^i = \frac{1}{2}\ln\left|\frac{\partial \Phi^k \circ \psi_i^{-1}}{\partial x}\right| = \frac{1}{2}\ln\left|\frac{\partial \Phi^k \circ \psi_i^{-1}}{\partial y}\right|$, and W is the Willmore energy[4].

Then our second application concerns the weak compactness of the conformal immersions with L^2-bounded second fundamental form. The following result was proved first in **[Ri14]** when the conformal class of the surface does not degenerate and has been extended to the general case of degenerating Riemann surfaces in **[KL]**. We shall present a different approach for proving this result as being a consequence of Theorem 31.

Theorem 33 (Laurain-Rivière 18, (8)) *Let Σ be a closed surface of genus strictly greater than 1 and $\Phi_k \in \mathcal{E}_\Sigma$ a sequence of weak immersion into $\mathbb{R}^N$ with L^2-bounded second fundamental form. Then, up to a subsequence, for any connected component σ of $\widetilde{\Sigma}$, the nodal surface of the converging sequence $(\Sigma, \Phi_k^*\xi)$, there exists a Möbius transformation Ξ_k of $\mathbb{R}^N$ such that*

$$\Xi_k \circ \Phi_k(\Sigma) \subset B(0,R)$$

where R depends only on N and there exists at most finitely many points $\{a_1, \dots, a_L\}$ of σ such that if we denote $\Psi_k = \Xi_k \circ \Phi_k \circ \phi_k$, then

$$\Psi_k \rightharpoonup \Psi \text{ weakly in } W^{2,2}_{loc}\left(\sigma \setminus \{a_1, \dots, a_L, q_1, \dots q_K\}, \tilde{h}\right),$$

where Ψ is a weak conformal (possibly branched) immersion of $(\sigma, \tilde{h})$ into $\mathbb{R}^N$ and the q_i are the punctures of $(\sigma, \tilde{h})$ and $\phi_k : \widetilde{\Sigma} \to \Sigma$ such $\phi_k^*(h_k) \to \tilde{h}$ in $C^\infty_{loc}(\widetilde{\Sigma})$.

Futhermore, for any compact $K \subset \sigma \setminus \{a_1, \dots, a_L, q_1, \dots q_K\}$ there exists $C_K > 0$ such that

$$\sup_{k \in \mathbb{N}} \| Log |d\Psi_k|_{\phi_k^* h_k} \|_{L^\infty(K)} \leq C_K,$$

where C_K depends only on m, K and the L^2-bound on the second fundamental form of Φ_k.

Here we consider the hyperbolic case, since in the sphere case the existence of a non compact conformal group is the additional difficulty already treated in **[MR]** and **[MR2]** and in a degenerating torus, the injectivity radius uniformly blow down.

Finally, we also proved that considering a sequence of Riemann surfaces with bounded total curvature and entropy (see below for precise definition) then we can find a finite conformal atlas in which the conformal factor is uniformly bounded. This result is in the spirit of Cheeger and Gromov **[CG]**, Trojanov **[Tro]** and most recently Shioya **[Shi]** for Riemannian surfaces. Indeed, we prove a general compactness result for sequence of metrics on a given closed surface assuming only that the area and the total curvature are uniformly bounded and that the entropy of the Gaussian curvature is also bounded. The first assumptions are the weakest possible. And the second is made to rule out a long thin cylinder closed by a two spherical cap, see **[Tro]** and reference therein for more examples of degenerating metrics with bounded curvature and area.

The entropy of the Gaussian curvature of a given metric is defined as follows : let Σ be a closed surface and h be a Riemannian metric with Gaussian curvature equal to

4. see Section 3 for a precise definition.

K_h, then we set

$$E(h)) = \int_\Sigma K_h^+ \ln(K_h^+)\, dv_h,$$

where $K_h^+ = \max(0, K_h)$ and we set $K_h^+ \ln(K_h^+) = 0$ when $K_g^+ = 0$. It was introduce by Hamilton in the context of Ricci flow on surfaces. He notably proved that it is monotonically increasing along the Ricci flow on spheres with positive curvature, see [**Ham**] and [**Chow**]. In order to apply directly our preceding result, we introduce a slightly stronger notion of entropy. Let Σ be a closed surface with a reference metric h_0, then we set

$$\tilde{E}_0(h) = \int_\Sigma \ln(e + |K_h dv_h|_{g_0})|K_h|\, dv_h.$$

Then, considering this notion of entropy, we get the following compactness result.

Theorem 34 *For any closed Riemannian surface (Σ, h_0) and any sequence of smooth metric h_k such that*

$$\int_\Sigma |K_{h_k}|\, dv_{h_k} + \tilde{E}_0(h_k) = O(1),$$

then for each component σ of the thick part of (Σ, h_k), then, up to a dilation of the metric by a factor e^{-C_k}, h_k converges weakly in $(L^\infty_{loc}(\sigma))^*$.

More precisely, up to a subsequence, one of the following occurs

- *(i)* genus$(\Sigma) = 0$, *then there exists C_k such that if $e^{-C_k} h_k = e^{2u_k} h_0$, where g_0 is the metric of the standard sphere, and u_k is uniformly bounded,*
- *(ii)* genus$(\Sigma) = 1$, *then up to a first dilation, (Σ, h_k) is isometric to $\mathbb{C}/(\mathbb{Z} \times v_k\mathbb{Z})$ where v_k lies in a fundamental domain of, then there exists C_k such that if $e^{-C_k} h_k = e^{2u_k}|dz|^2$ then u_k is bounded in $L^\infty_{loc}(\mathbb{C}/(\mathbb{Z} \times v_k\mathbb{Z}))$.*
- *(iii)* genus$(\Sigma) \geq 1$, *then let σ be a connected component σ of the nodal surface of (Σ, h_k), then there exists C_k such that if $e^{-C_k} h_k = e^{2u_k} h_k^y$, where h_k^y is the corresponding hyperbolic metric, and u_k is bounded in $L^\infty_{loc}(\sigma)$.*

4.4 A brief introduction to Willmore surfaces

The Willmore energy was introduced in the XIX-th century in non-linear mechanics as being the *ad hoc* modelization of the free energy of a bent two dimensional elastic membrane. It was then independently introduced in geometry by Blaschke around 1920 in an effort to merge minimal surfaces theory and conformal invariance, see [**Bla**]. If Φ denotes the immersion of an abstract closed surface Σ into a euclidian space $\mathbb{R}^N$, the Willmore energy of such an immersion is given by

$$W(\Phi) := \int_\Sigma |\vec{H}_\Phi|^2\, dvol_{g_\Phi}$$

where g_Φ is the first fundamental form of the immersion (*i.e.* the induced metric by Φ), $dvol_{g_\Phi}$ is the associated volume form and $\vec{H}_\Phi := 2^{-1} \operatorname{tr}_{g_\Phi} \vec{\mathbb{I}}_\Phi$ is the half of the trace of the second fundamental form $\vec{\mathbb{I}}_\Phi$ of Φ. Blaschke proved that the lagrangian W is invariant under conformal transformations for closed surfaces, see also section 7.3 of [**Will**]. That is to say, for any generic [5] element Ψ in $\mathcal{M}(\mathbb{R}^N \cup \{\infty\})$, the Möbius

5. "Generic" means that $\Psi^{-1}(\infty) \cap \Phi(\Sigma) = \emptyset$.

group of conformal transformations of $\mathbb{R}^N \cup \{\infty\} \simeq S^N$

$$W(\Psi \circ \Phi) = W(\Phi) \quad .$$

During a long time minimal surfaces and their conformal transformations were the only known critical points of W. One of the reasons for such a lack of examples and progress during almost 45 years is possibly due to the fact that the Euler-Lagrange equation of W is a non-linear elliptic system of order 4 which made it difficult to be studied from an analyst perspective at a time where this high order PDE theory was not much developed [6]. The seminal paper of Willmore, see [**Wi**], relaunched the interest for the lagrangian to which his name has since then been associated. Let us give the most important results of the theory.

In 1965, Willmore proved that the round sphere is the absolute minimizer of the Willmore energy in any codimension, and he formulated the so-called *it Willmore conjecture* that the Clifford torus must be the minimizer among the space of surfaces of genus bigger or equal than 1.

In 1982, Li and Yau (see [**LiYau**]) made an important link between the conformal volume and the Willmore energy. This work contains as by product some special case of the Willmore conjecture and the fact that every surface with Willmore energy strictly below 8π must be embedded.

In 1984, Bryant proved the existence of an holomorphic quartic associated to each Willmore surface. In order to do so he gave a precise correspondence between Willmore surfaces and minimal surfaces into de Sitter space. This can be seen as a generalization of the correspondence between CMC surfaces and harmonic maps into S^2, see chapter 2 of [**Hel2**]. In particular he gave a complete classification of Willmore spheres in $\mathbb{R}^3$, proving that they are inversion of minimal surfaces in $\mathbb{R}^3$ with planar ends. This work have been later generalized by Montiel to dimension 4, see[**Mon**]. My student, Marque, has recently made a review of the results coming from Bryant's technics and has generalized the correspondence to constrained Willmore surfaces and conformally CMC surfaces, see [**Mar1**].

In 2000, Ros solved the Willmore conjecture for tori invariant under the antipodal map, see [**Ros**].

In 2004, Kuwert and Schätzle proved the first ε-regularity result for the Willmore equation (see [**KuS**]). Here, let us remind the classical Willmore equation. If Σ is a Willmore surface then we have

$$\Delta_\Sigma H + H|\mathring{A}|^2 = 0.$$

6. Indeed, in a conformal parametrization Φ, the Willmore functional may be recast as

$$W(\Phi) = \frac{1}{4}\int_\Sigma |\Delta_{g_\Phi}\Phi|^2 \, dvol_{g_\Phi} \quad ,$$

thereby giving rise to a fourth-order problem

From the analytic point of view this equation is very hard to study, since first of all the non-linearity is cubic in the curvature and *a priori* not even in L^1 and secondly the coefficients of the elliptic operator depends itself of the immersion. Kuwert also proved with Bauer, continuing the work of Simon [**Simon**], that the infimum of the Willmore energy is achieved for all genus, [**BK**].

In 2008, Rivière also proved the ε-regularity and the existence of minimizer, this time in any codimension (see [**Riv08**]). In order to do so, he exhibited conservation laws, which permitted him to apply some compactness by compensation theory. Later Bernard proved that those conservation laws are in fact by product of Noether's theorem, see [**Ber**]. Let us remind those conservation laws, in codimension 1. A conformal immersion $\Phi : \mathbb{D} \to \mathbb{R}^3$, is Willmore if and only if

$$\mathrm{div}(2\nabla \vec{H} - 3H\nabla\vec{n} + \vec{H} \wedge \nabla^\perp \vec{n}) = 0$$

where $\vec{n}$ is the normal. This conservation law comes from the invariance by translations. Using the invariance by rotation and dilation we get the existence of two quantities $\vec{R}$ and S satisfying the following system

$$\begin{cases} \Delta S = -\nabla\vec{n}.\nabla^\perp\vec{R} \\ \Delta\vec{R} = \nabla\vec{n}.\nabla^\perp\vec{R} + \nabla^\perp S.\nabla\vec{n} \\ \Delta\Phi = \frac{1}{2}(\nabla^\perp S.\nabla\Phi + \nabla^\perp\vec{R} \wedge \nabla\Phi) \end{cases}$$

Thanks to the jacobian structure of the right hand side, the regularity of $\vec{R}$ and S can be improved. Then we can transfert it to Φ thanks to the equation

$$\Delta\Phi = \nabla\vec{R}\nabla^\perp\Phi + \nabla S\nabla^\perp\Phi.$$

In fact this is not totally exact, since in order to transfer some estimates on $\vec{R}$ and S to Φ, we already need to control $\nabla\Phi$, that is to say the conformal factor. This is not surprising since the equation must be invariant by conformal transformation. Hence before starting to work we have to choose a good representation of the surface (a kind of gauge) in which the conformal factor is controlled. It is always possible when the conformal class is bounded, as already noted by Müller and Šverák, see [**MuSv**], see also Chapter 5 of [**He**]. In fact as soon as the total curvature is in L^2, we can always locally find a Coulomb frame $(\vec{e}_1, \vec{e}_2)$ satisfying

$$\mathrm{div}(\langle \vec{e}_1, \nabla\vec{e}_2\rangle) = 0$$

and

$$\int_{\mathbb{D}} |\nabla\vec{e}_1|^2 + |\nabla\vec{e}_2|^2\,dx \le C\int_{\mathbb{D}} |\nabla\vec{n}|^2\,dx.$$

Moreover, the conformal factor λ, defined by $e^{2\lambda}dx^2 = \Phi^*(\xi)$ satisfies

$$\Delta\lambda = \nabla^\perp\vec{e}_1.\nabla\vec{e}_2,$$

and the Wente inequality insures that if λ is bounded at 0 then it is bounded in any compact set of the disc.

In 2012, Marques and Neves proved the Willmore conjecture, see [**MN**]. In order to do so they revisited the min-max theory of Almgren-Pitts, which has given rises to very important developments up to the recent proof of the Yau conjecture by Song (see [**Son**]).

The Willmore conjecture in codimension bigger than 2 and in genus bigger than 2 is still open. In fact we suspect that the Lawson examples, see [**Law**] and [**BWW**], should be minimizers, because they are the most symmetric minimal surfaces we know in S^3, but it seems that we are still far of this goal.

Beyond the description of minimizers, another direction of research consist in study the compactness of critical points and more generally of Palais-Smale sequences. The main result of the next section is a first step in this direction, see also [**BR3**]. Indeed, such a description will make a very strong tool to study either the space of immersions or even the Teichmüller space. In fact the goal would be to consider the Willmore functional as a Morse function on those space. For the space of immersion we would refer to [**Mich**] and also the promising approach of Rivière [**Riv15**] which should lead to the 16π-conjecture, *i.e.* the minimal energy in order to perform a sphere inversion is the one given by the first non-trivial Bryant sphere. For the Teichmüller, space one should consider the constrained Willmore problem, *i.e.* the minimizer of the Willmore functional in a given conformal class. The study of the minimizer of the constrained Willmore problem is currently developed by the German school of integrable system, see [**HP**] and reference therein. We believe that our methods would allow us to prove some kind of Palais-Smale condition and then to develop a totally variational approach in the spirit of the one developed for minimal surfaces in [**Moore**] for instance.

4.5 Quantification of the energy of Willmore surfaces : the general case

In the present paragraph we are interested in sequences of immersions Φ_k of a given closed surface Σ into $\mathbb{R}^N$ which are critical points of W and which are below a given energy level. It was proved in [**Riv08**] and [**Ri14**] that critical points to W satisfy an ε-regularity property and that, under the assumption that the conformal class of the metrics g_{Φ_k} is controlled, the sequence is compact in any C^l norm away from finitely many points (modulo the action of the Möbius group $\mathcal{M}(\mathbb{R}^m \cup \{\infty\})$). The iteration of this result at the various concentration points generates a bubble tree of Willmore surfaces. Then, notably using Lemma 21 of Section 3.4, Bernard and Rivière proved an energy identity when the conformal class is bounded. The main goal of this section is to treat the general case.

In Section 4.3, we give an estimate on the Green function of the Laplace operator which permits to extend the previously mentioned concentration compactness result for Willmore surfaces to the case where the underlying conformal classes degenerate. The following theorem, which is the main result of this section, asserts that, if there is some loss of energy in a collar region, then the amount of this loss has an explicit expression in terms of a residue and the hyperbolic length of the collar. It can be seen as a counterpart of a result of Zhu [**Zhu**] for harmonic maps. Let us define this residue. In fact it comes as integration of the three following conservation laws given respectively by the invariance by translation, rotation and dilation :

$$
\begin{aligned}
&\operatorname{div}(\nabla\vec{H} - 3\pi_{\vec{n}}(\nabla\vec{H}) - \star(\nabla^{\perp}\vec{n}\wedge\vec{H}) = 0\\
&\operatorname{div}(\langle\nabla\vec{H} - 3\pi_{\vec{n}}(\nabla\vec{H}) - \star(\nabla^{\perp}\vec{n}\wedge\vec{H},\Phi\rangle) = 0\\
&\operatorname{div}((\nabla\vec{H} - 3\pi_{\vec{n}}(\nabla\vec{H}) - \star(\nabla^{\perp}\vec{n}\wedge\vec{H})\wedge\Phi - (-1)^{N-1}2(\star(\vec{n}\llcorner\vec{H}))\llcorner\nabla^{\perp}\Phi) = 0
\end{aligned}
\tag{4.6}
$$

Without loss of generality we can assume that the collar region is conformally parametrized by $\mathbb{D}\setminus B(0,e^{-1/l_k})$ where l_k is the length of shrinking geodesic which corresponds to the circle of radius $e^{-1/2l_k}$. For $\Phi_k : \mathbb{D}\setminus B(0,e^{-1/l_k}) \to \mathbb{R}^m$ a Willmore immersion, for $e^{-1/l_k} < r < 1$, we set

$$
\begin{aligned}
2\pi\vec{c}^{\,k} &= \int_{\partial B(0,r)} \partial_\nu\vec{H}_k - 3\pi_{\vec{n}_k}(\partial_\nu\vec{H}_k) - \star(\partial_\tau\vec{n}_k\wedge\vec{H}_k)\,d\sigma,\\
2\pi c_0^k &= \int_{\partial B(0,r)} -\langle(\partial_{\nu_k}\vec{H}_k - 3\pi_{\vec{n}_k}(\partial_{\nu_k}\vec{H}_k) - \star(\partial_{\tau_k}\vec{n}\wedge\vec{H})),\Phi_k\rangle\,d\sigma,\\
2\pi\vec{c}_1^{\,k} &= \int_{\partial B(0,r)} -(\partial_{\nu_k}\vec{H}_k - 3\pi_{\vec{n}_k}(\partial_{\nu_k}\vec{H}_k) - \star(\partial_{\tau_k}\vec{n}\wedge\vec{H}_k))\wedge\Phi_k\\
&\qquad - (-1)^{N-1}2(\star(\vec{n}_k\llcorner\vec{H}_k))\llcorner\partial_{\tau_k}\Phi_k\,d\sigma.
\end{aligned}
\tag{4.7}
$$

where ν and τ are respectively a unit normal and a unit tangent to $\partial B(0,r)$ such that (ν,τ) is oriented, and $\vec{n}_k$ and $\vec{H}_k$ are respectively the normal $(m-2)$-vector and the mean curvature vector of Φ_k. The operations $\star$, $\llcorner$ and $\wedge$ are classical operations on multi-vectors (see symbol section for precise definitions).

The quantities $\vec{c}$, c_0 and $\vec{c}_1$ are independent of r thanks to the conservation laws. The quantity $\vec{c}$ already appeared in **[Riv08]** and **[BR2]** as a residue which permits to erase true branch point of Willmore punctured disks (see also **[KS]**). But it is not conformally invariant, even not scaling invariant. For instance $\vec{c}$ is zero when computed on the closed geodesic of a catenoid, but not on the corresponding curve on the inverted catenoid, see Remark 1.1 **[BR2]**. This remark has a very important consequence in the bubbling phenomena, see paragraph below and Theorem 36. Moreover, it is hopeless to try to find a residue, like $\vec{c}$, which measures a defect of regularity and will be conformally invariant. Taking again the example of the inverted catenoid, we can blow-up it to a union of two planes, if we get a scaling invariant residue, it must vanishes on the inverted catenoid, since the blow-up is smooth. Hence this residue would not detect the defect of regularity of the inveted catenoid.

But c_0 and $\vec{c}_1$ are clearly invariant under the composition by isometries and dilations. They can be considered as the Willmore analogue of flux for CMC-surfaces (see **[Lo]**). Since the Willmore equation has fourth order, it is not surprising to get two "fluxes". Moreover, we can check (see appendix of **(7)** for details) that there exists Willmore surfaces for which those residues are non zero. Such examples are provided by considering Willmore Hopf tori, see **[Pinkall]**. For Hopf tori, the cancellation of the residue $\vec{c}_1$ forces the generating curve (an elastica on S^2) to be a geodesic of S^2, hence the surface will be equivalent to a Clifford torus. Therefore the family of Hopf tori produced by Pinkall, in the above mentioned work, provides good examples. More generally, it is easy to prove that those residues vanish on any minimal surfaces of S^N. Hence it provides a new way to detect Willmore surfaces which are not conformal to a minimal surface.

Those quantities being defined we can state the main result of this section.

Theorem 35 (Laurain-Rivière 18, (7)) *Let (Σ, h_k) a sequence of closed surfaces with fixed genus, constant curvature and normalized volume if needed. We assume that this sequence converges*[7] *to a nodal surface $(\widetilde{\Sigma}, \widetilde{h})$ and we denote by $\{\gamma_k^i\}$ the finite number of pinching geodesics. Then let $\Phi_k : (\Sigma, h_k) \to \mathbb{R}^m$ a sequence of conformal Willmore immersions with bounded energy, i.e.*

$$\limsup_{k\to+\infty} W(\Phi_k) < +\infty$$

and such that, around every degenerating geodesic,

$$\lim_{k\to+\infty} \frac{\vec{c}_1^{\,k}}{\sqrt{l_k}} = 0.$$

Then, denoting by $(\tilde{\Sigma}^l)_{1\leq l\leq q}$ the connected components of $\tilde{\Sigma}$, there exists q branched smooth immersions $\Phi^l_\infty : \tilde{\Sigma}^l \to \mathbb{R}^m$ and a finite number of possibly branched immersions $\omega_j : S^2 \to \mathbb{R}^m$ and $\zeta_t : S^2 \to \mathbb{R}^m$ which are all Willmore away from possibly finitely many points, and such that, up to a subsequence,

$$\lim_{k\to+\infty} W(\Phi_k) = \sum_{l=1}^{q} W(\Phi^l_\infty) + \sum_{j=1}^{p} W(\omega_j) + \sum_{t=1}^{q} (W(\zeta_t) - m_t 4\pi) \quad . \tag{4.8}$$

where m_t is the integer multiplicity of ζ_t at the origin.

In [**BR**], Bernard and Rivière established the corresponding result but under the additional assumption that the conformal class induced by the sequence was pre-compact in the moduli space. We observe that, under this much stronger assumptions, the branched immersions ω_j and ζ_t are "true" Willmore surfaces in the sense that the Willmore equation is satisfied everywhere away from the branched points and moreover the first residue, $\vec{c}$, is zero on any curve surrounding these branched points. This excludes surfaces like the catenoid (or its inversion) in the bubble tree, but not the Enneper surface for instance. This observation is a starting point for improving the classical bound of 8π, the one which insures strong compactness, see Theorem 35. If the conformal class degenerates we cannot exclude *a priori* the first residue to be non zero around the cuspidal point for both the ω_j and the ζ_t, since the curve on which we compute the integral is not homologous to a point. However, we conjecture that in view of previous work on harmonic maps [**LLW**], that the condition on the residue can be dropped under some natural stability conditions, more precisely

Conjecture 4

1. *Any sequence of Willmore surfaces with bounded energy and bounded index satisfies an energy identity.*
2. *Any sequence of L^2-bounded second fundamental form produced by some finite dimensional min-max method satisfy an energy identity.*

The previous remark which excludes the catenoid in the bubble trees, permits us to increase the level under which compactness holds true (modulo the action of the Möbius group).

7. The convergence holds in the classical sense of Mumford compactification recalled in [**Hum**].

Theorem 36 *Let Σ be a closed surface and $\Phi_k : \Sigma \to \mathbb{R}^3$ a sequence of conformal Willmore immersions such that $[\Phi_k^*(\xi)]$, the conformal class of the pullback metric, remains in a compact set of the moduli space and*

$$\limsup_{k\to+\infty} W(\Phi_k) < 12\pi.$$

Then, there exists a diffeomorphism ψ_k of Σ and an conformal transformation Θ_k of $\mathbb{R}^3 \cup \{\infty\}$, such that $\Theta_k \circ \Phi_k \circ \psi_k$, converges (up to a subsequence) to a smooth Willmore immersion $\Phi_\infty : \Sigma \to \mathbb{R}^3$ in $C^\infty(\Sigma)$.

This result was already known, when $\Sigma = S^2$, in fact there is a complete classification of Willmore sphere in $\mathbb{R}^3$, see **[Bry]**. It is also known that we have compactness when Φ_k is an embedding, see **[Li]**. But nothing, was known when Φ_k is an immersion with energy above 8π. Recently, my student N. Marque proved that in fact the inequality is large for tori, **[Mar3]** and gave some credit to the following conjecture

Conjecture 5 *Sequences of Willmore surfaces with bounded conformal class and energy strictly below 16π are compact.*

However, paying more attention to the Bryant examples, he also proved that the space of Willmore spheres of energy 16π is not compact.

In fact, looking at the proof of Theorem 36 we can be more precise. Either we blow a compact Willmore surface and in this case the blow-up requires at least 12π of energy

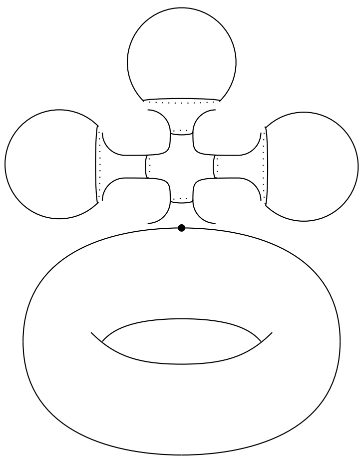

FIGURE 4.2 – We must glue at least 12π...

or we blow a non-compact Willmore surfaces (like the Enneper surface),

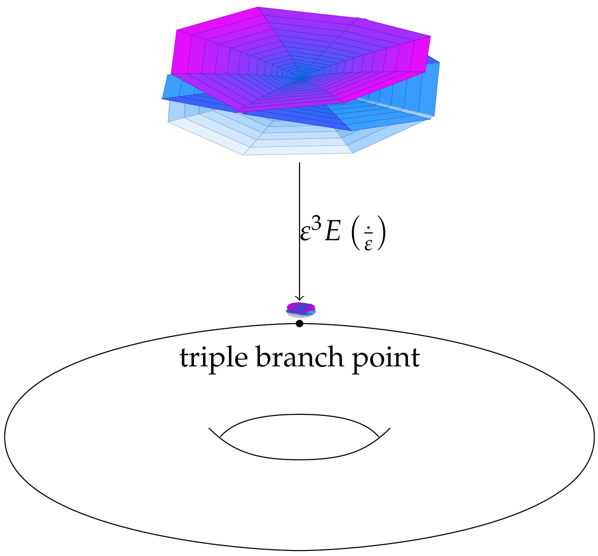

FIGURE 4.3 – ... or to have a branch point.

and the limit surface will have at least one branch point of order three and its energy should be at least 12π. Therefore, if we exclude the possibility for the limit surface to have branch points, we will get compactness below $\beta_g + 12\pi$, where

$$\beta_g = \inf\{W(\Phi) \,|\, \Phi : \Sigma \to \mathbb{R}^3 \text{immersion and } genus(\Sigma) = g\}.$$

Coming back to Theorem 35, the new difficulty posed by the non compactness of the underlying conformal classes in comparison with the previous quantization result in [**BR**] comes from the formation of *collars*. By definition, those *collars* are conformally equivalent to degenerating annuli. Unlike the *neck regions* from [**BR**], which are also conformally equivalent to degenerating annuli, the solution is **non extendable** throughout the interior disc of the annuli. This lack of extendability is responsible for the presence of residues which were automatically zero in the Bernard-Rivière case. The main achievement of this work is to obtain a control of these *residues*. We need in fact very precise estimates since when we integrate conservation laws on annuli, those residues correspond to the coefficient in front a log function whose derivatives have a bad behaviour in L^2. Finally, as already explained, let me underline that the control of the conformal factor plays a crucial role in the analysis of the Willmore equation, hence Theorem 32 is essential in our proof.

In [**Zhu**], Miaomiao Zhu proved that for general sequences of harmonic maps from degenerating Riemann surfaces into a given manifold possessing *harmonic spheres* [8] "energy quantization" usually **does not** hold. Therefore our result above comes as a little surprise, since we expected two residues for a fourth order problem, but we were able to prove that if there is some loss it can come only from the second residue $\vec{c}_1$.

A natural extension of this work would be to consider Palais-Smale sequences. But the counter example to the quantization of the energy of Parker [**Pa**] in the setting of

8. Harmonic spheres in a manifold N^n are non constant harmonic maps from S^2 into N^n. This space is non empty for instance if $\pi_2(N^n) \neq 0$ but this condition is not necessary as the example $N^n = S^3$ shows.

harmonic maps brings us naturally to the idea that this should not hold to be true for Willmore in general. Nevertheless, regarding applications, the no-neck energy was established for Palais-Smale sequences issued from min-max sequences in a viscosity approach in the sphere case, see [**Riv15**], with the result of proving that the cost of the sphere inversion is *a priori* a multiple of 4π, see also [**MiR**]. Our work can also be considered as the first step to an energy identity for min-max sequences for arbitrary genus. In fact it will be consider in an incoming work, we aim to prove that the residues indeed vanishes in the special case of a min-max sequences.

4.6 Gluing technics for Willmore Surfaces

This section is based on some work in progress with J. Lira.

Despite the important work done during the last decade on Willmore surfaces. We still have very few examples. Indeed the study of the moduli space of Willmore surface is two-sided. The first one consists in the one developed in the previous paragraph studying *a priori* the behaviour of sequences of Willmore immersions. The second consists in finding concrete example where this behaviour really occurs. The space of spheres is nicely described by the work of Bryant. For tori, Pinkall produced some non-minimal example. But for higher genus almost nothing is known. However, our main result with Rivière permits to measure the loss of energy by an explicit residue when the conformal class degenerate. It is a natural question to try to produce those examples. That is to say Willmore surfaces with bounded energy and degenerating conformal class.

For the Hopf tori produce by Pinkall, the study reduces to the space of elastica in S^2, see [**Heller**]. It has to be noted, that the German school has developed an integrable system approach to produce example of Willmore (constrained) surfaces, see [**BPP**] and references therein.

We are interested in the higher genus case where a collar region appears. For instance, it will be of great interest to produce a sequence of Willmore surfaces of genus 2 whose conformal class degenerates to a nodal surface consisting of the union of two genus 1 surfaces. Looking back to the the analysis of the moduli space of compact CMC immersions, we know that a real breakthrough came with the work of Kapouleas [**Kapouleas**], in particular with the appearance of his gluing method.

Hence the idea to produce the first natural example of Willmore surfaces with degenerating conformal class seems to glue two Clifford tori. More precisely, we consider one Clifford tori T_1 and an inverted copy T_2, then by contracting T_2 enough it will look like a plane with a tiny handle and we could try to glue it along a annular region which will generate the collar.

Of course this is an ambitious. And we proceed by steps. Currently we glue two non degenerate disks, as Mazzeo, Pacard and Pollack in the CMC case, see [**MPa**]. More precisely we consider one disk and a inverted one. Mazzeo, Pacard and Pollack perform the gluing by matching some Neumann data along some neck. We prefer to use the method described by Zolotareva, see [**Zo**], inspired by the work of Mazzeo

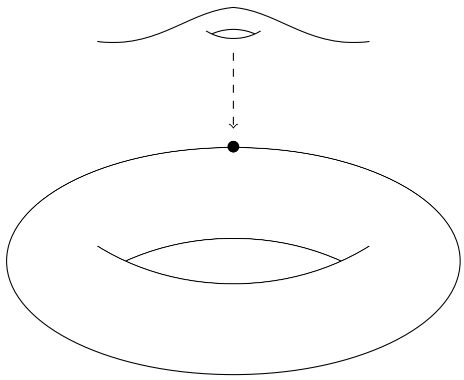

FIGURE 4.4 – Gluing of two Clifford tori

Pollack and Uhlenbeck in the Yamabe case, see [**MPU**], where she first constructs an approximate solution thanks to some cut-off function and then she makes a careful study of the linearized operator in the "gluing region". The control of the kernel is essential in this kind of analysis. That is the reason why we start with non-degenerate disks, we can focus only on the "gluing region". Since this region correctly rescaled converges to a punctured plane, the linearized operator become Δ^2 and we must control precisely the decreasing of our solution, to get a Kernel which can be killed by a right choice on our free parameters, here the conformal group, hence we can apply the implicit function theorem. This is the so called finite-reduction method. In a second time, we will glue and inverted disc to a full Clifford torus. We could chose any Willmore tori, but since the Clifford one is in some sense the simplest one, the study of the kernel operator should be easier. Here the main difficulty, is that we will have less freedom since one the component must be closed. Finally, we hope to take advantage of the previous analysis to glue two Clifford tori with degenerating conformal class as in the above picture.

Chapitre 5

Willmore surfaces in asymptotic settings

This chapter is based on the following two articles : **(11)** and **(12)**.

In this chapter, I explain how the methods developed in my PhD for constant mean curvature (CMC) surfaces, was generalized to Willmore surfaces. In the preceding chapters, we have presented some weak-compactness properties or energy identities. This is the first step to study concentration phenomena. But if one desires to get some more information, like locus of concentration or even the absence of concentration, energy estimates are not sufficient and we need some pointwise estimates.

Coming back to our scalar model, Yamabe-type equations, a deep pointwise analysis was done during the first decade of this century. It permitted to give a complete answer to the question of the compactness of metrics with constant scalar curvature. The strategy was initiated by Schoen in the late 80's when he formulated the compactness conjecture. The main idea consists in two steps. First, we need to prove that a *a priori* non-compact sequence has finitely many concentration points where its profile is like the one of the Green function of the conformal Laplacian. Then, applying the Schoen-Pohožaev formula, we can derive a contradiction thanks to the positive mass theorem. In fact when the dimension become large, the mass is not well defined and there is even some examples of non-compact sequence of solution. We refer to [**BM**] for a complete overview on this problem.

During my PhD, I developed a similar analysis for the CMC equation. The initial goal was to characterize the locus of concentration of small CMC surfaces into a Riemannian manifold. Despite the fact that the CMC equation is, like the Yamabe-equation, critical, I had to develop new tools. Especially for the first step, the blow-up analysis. Indeed, the CMC equation is a strongly coupled system, hence we have no maximum principle. I have replaced, the maximum principle, by a careful analysis of the linearized operator in order to get pointwise estimates. The second step, the use of Pohožaev formula, should have been replaced by its equivalent for the CMC surfaces, namely the flux formula. But as we have seen in chapter 2, we can derive the Pohožaev formula by testing the equation against a solution of the linearized operator. I noticed that we can also recover the same information than the one contained in the flux formula, testing the equation against appropriate solutions of the linearized operator. This method is much more robust, since it can also be apply when one has no access to this kind of formula, which is the case for Willmore surfaces as we will see later.

Before speaking of the kind of pointwise blow-up analysis that we have developed for Willmore surfaces, I will present a baby-case in order to give some ideas without drowning it into technicalities.

5.1 A baby case

In this section we present a very simplified proof of the following claim.

Claim : Let (M, g) a 3-Riemannian manifold and Σ_n a sequence of CMC surface with mean curvature $H_n \to +\infty$ and such that

$$H_n\Sigma_n \text{ converge to a round sphere.}$$

Then Σ_n concentrates to a critical point of the scalar curvature.

In fact this situation really occurs, see **[Ye]**. The case of multiple spheres is also possible, see **[Zo]**, and the necessary condition was one part of my PhD. Whether it was the sufficient or the necessary condition, the multiple spheres case is much more involved since it requires a very careful study of the neck regions.

Step 1 : adjustment of the bubble.

Up to a subsequence, Σ_n converge to a point p, we choose a chart a round the center of mass of Σ_n.

We parametrize $H_n\Sigma_n$ by a conformal map $u^n : S^2 \to \mathbb{R}^3$. By hypothesis, it converges to a round sphere parametrized by ω and it satisfies, in any conformal chart, the following equation

$$\Delta u^n = -2u^n_x \wedge u^n_y + E^n,$$

where E^n is a small term which come from the fact that the metric is almost euclidean. Indeed, we have rescaled everything by a factor H_n around p. Of course, u^n is close to ω, but to what extent can we deform ω to solve exactly the equation ?

The first term in the expansion of E_n is given [1] by

$$E_n = \frac{c}{H_n^2}\mathrm{Ric}_{ij}(p)\omega^i + \dots$$

and in fact we can solve it explicitly by setting

$$u^n = \omega + \frac{c'}{H_n^2}\mathrm{Ric}_{ij}(p)\omega^i + \dots = \omega^n + \dots.$$

1. the expansion is simplified for the sake a simplicity

Futhermore, by applying an isometry of $\mathbb{R}^3$ or a conformal transformation of S^2 on ω^n, we can assume that

$$u^n(0) = \omega^n(0), \nabla u^n(0) = \nabla \omega^n(0)$$

and

$$|\nabla u^n| \text{ and } |\omega^n| \text{ are both maximal at } 0.$$

If we consider the following expansion for u^n,

$$u^n = \omega^n + r^n,$$

where r^n is called the reminder, it satisfies

$$r^n(0) = \nabla r^n(0) = \langle \nabla^2 r^n, \nabla \omega^n \rangle = 0. \tag{5.1}$$

Step 2 : Estimate on the reminder.

Now if we expand the equation, the reminder satisfies the linearized equation [2] of the euclidean equation plus a new error term, namely

$$\Delta r^n + 2(r^n_x \wedge \omega_y + \omega_x \wedge r^n_y) = \frac{c}{H_n^3} Ric_{ij,k}\omega^i \omega^k + \dots \tag{5.2}$$

The main goal of this step is to prove that the reminder behaves like the geometry, that is to say

$$|\nabla r^n| = O\left(\frac{1}{H_n^3}\right). \tag{5.3}$$

This is quite natural, since if the reminder were much bigger than the geometry, everything would have to occur as if we were in the euclidean space. But we know that the round spheres are the only CMC spheres in this situation, so $r_n = cste$ and we reach a contradiction. This heuristic argument is formalized by the study of the linearized operator. In fact let $x_n \in S^2$ a point where $|\nabla r^n|$ achieves its maximum. Then we rescale r^n in a chart which contains x_n and 0. We have

$$\frac{r^n}{\|\nabla r^n\|_\infty} \to r^\infty$$

where r^∞ is a non trivial solution of

$$\Delta r + 2(r_x \wedge \omega_y + \omega_x \wedge r_y) = 0.$$

But we know, by Proposition A.2 of **(b)**, that the only solutions of the linearized equation which satisfies (5.1) with appropriate decay at infinity are constant. Hence we reach a contradiction and prove (5.3).

This argument is very general, it works as soon as one can prove that the only solutions of the linearized operator are the one generated by the invariance group. Then it suffices to kill those solutions by adjusting correctly all the parameters of this invariance group to get the optimal estimate. Namely, the reminder is controlled by

2. We should also linearized the conformal condition, but we do not to it to simplify.

the geometry.

Step 3 : The flux formula through the linearized equation.

Thanks to (5.3), we can multiply (5.2) by H_n^3 and pass to the limit, which gives

$$\Delta r^\infty + 2(r_x^\infty \wedge \omega_y + \omega_x \wedge r_y^\infty) = cRic_{ij,k}\omega^i\omega^k.$$

Then we consider the three solutions of the linearized operator generated by translations or dilatations of the parametrization, namely : ω_x, ω_y and $x\omega_x + y\omega_y$. We multiply the previous equation by those solutions and we integrate by part. Since the operator is self adjoint then the left hand side vanishes and on the right hand side, thanks to Bianchi formula, we get

$$0 = \nabla S(p),$$

where S is the scalar curvature, which proves the claim. □

Of course, everything here was simplified a lot to highlight the main ideas. But this analysis needs to be made very carefully, especially with respect to the convergence and the estimates on the decay of limiting solutions. But it has the main advantage to be very robust and could be apply in many context where the maximum principle is absent. It is especially well adapted to conformally invariant problems.

5.2 Small Willmore surfaces

Let Σ be a closed two dimensional surface and (M, g) a 3-dimensional Riemannian manifold. Given a smooth immersion $\Phi : \Sigma \hookrightarrow M$, the Willmore energy of Φ is defined as follows

$$W(\Phi) := \int_\Sigma H^2 \, dvol_{\bar{g}}, \tag{5.4}$$

where $\bar{g} := \Phi^*(g)$ is the pullback metric on Σ.

If the ambient manifold is the euclidean 3-dimensional space, we refer to section 4.4 for a brief introduction. We would like to underline that the investigation of the Willmore functional in non constantly curved Riemannian manifolds is a much more recent topic started in [**Mon1**, **Mon2**] where Mondino studied existence and non existence of Willmore surfaces in a perturbative setting.

Smooth minimizers of the L^2-norm of the second fundamental form among spheres in compact Riemannian three manifolds were obtained by Kuwert, Mondino and Schygulla in [**KMS**] where the full regularity theory for minimizers was settled taking inspiration from the approach of Simon [**Si**].

Let us also mention the work of Mondino and Rivière [**MR**, **MR2**] where, using the "parametric approach" inspired by the Euclidean theory of Rivière, they developed the necessary tools for studying the calculus of variations of the Willmore functional in Riemannian manifolds (*i.e.* the definition of weak objects and related compactness and regularity issues) are settled together with applications ; in particular the existence and regularity of Willmore spheres in homotopy classes is established.

Since, as usual in the calculus of variations, the existence results are obtained by quite general methods and do not describe the minimizing object, the purpose of the present section is to investigate the geometric properties of the critical points of W.

More precisely we investigate the following natural questions : let $\Phi_k : S^2 \hookrightarrow M$ be a sequence of smooth critical points of the Willmore functional W (or more generally we will also consider critical points under area constraint) converging to a point $\bar{p} \in M$ in Hausdorff distance sense ; what can we say about Φ_k ? are they becoming more and more round ? Has the limit point $\bar{p}$ some special geometric property ?

Before the description of the state of art and the new results in this direction, let us recall that a critical point of the Willmore functional is called a *Willmore surface* and it satisfies :

$$\Delta_{\bar{g}} H + H|\mathring{A}|^2 + H\mathrm{Ric}(\vec{n}, \vec{n}) = 0, \tag{5.5}$$

where $\Delta_{\bar{g}}$ is the Laplace-Beltrami operator corresponding to the metric $\bar{g}$, $(\mathring{A})_{ij} := A_{ij} - H\bar{g}_{ij}$ is the trace-free second fundamental form, $\vec{n}$ is a normal unit vector to Φ, and Ric is the Ricci tensor of the ambient manifold (M, g).

Throughout this section we will consider more generally *area-constrained Willmore surfaces*, *i.e.* critical points of the Willmore functional under area constraint ; the immersion Φ is an area-constrained Willmore surface if and only if it satisfies

$$\Delta_{\bar{g}} H + H|\mathring{A}|^2 + H\mathrm{Ric}(\vec{n}, \vec{n}) = \lambda H, \tag{5.6}$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier.

The first result in the direction of the above questions was achieved by Mondino in [**Mon1**] where it was proved that if (Φ_k) is a sequence of Willmore surfaces obtained as normal graphs over shrinking geodesic spheres centered at a point $\bar{p}$, then the scalar curvature at $\bar{p}$ must vanish : $\mathrm{Scal}(\bar{p}) = 0$.

In the subsequent papers [**LM1**, **LM2**] (see also [**LMS1**] and [**IMM1**]), Lamm and Metzger proved that if $\Phi_k : S^2 \hookrightarrow M$ is a sequence of area-constrained Willmore surfaces converging to a point $\bar{p}$ in Hausdorff distance sense and such that [3]

$$W(\Phi_k) \leq 4\pi + \varepsilon \quad \text{for some } \varepsilon > 0 \text{ small enough}, \tag{5.7}$$

then $\nabla \mathrm{Scal}(\bar{p}) = 0$ and, up to subsequences, Φ_k is $W^{2,2}$-asymptotic to a geodesic sphere centered at $\bar{p}$. Moreover, in [**LM2**], using the regularity theory developed in [**KMS**], they showed that if (M, g) is any compact Riemannian 3-manifold and a_k is any sequence of positive real numbers such that $a_k \downarrow 0$ then there exists a smooth

3. notice that the normalization of the Willmore functional used in [**LM1**, **LM2**] differs from our convention by a factor 2

minimizer Φ_k of W under the area-constraint $\mathcal{A}(\Phi_k) = a_k$. Futhermore, such sequence (Φ_k) satisfies (5.7) and therefore it $W^{2,2}$-converges to a round critical point of the scalar curvature. Let us mention that the existence of area-constrained Willmore spheres was generalized in **[MR2]** to any value of the area.

The main achievement of **[L12]** is the improvement of the perturbative bound (5.7) above to the global bound

$$\limsup_k W(\Phi_k) < 8\pi. \tag{5.8}$$

Secondly we improve the $W^{2,2}$-convergence above to *smooth* convergence towards a *round* critical point of the scalar curvature, *i.e.* we show that if we rescale (M, g) around $\bar{p}$ in such a way that the sequence of surfaces has fixed area equal to one, then the sequence converges smoothly, up to subsequences, to a round sphere centered at $\bar{p}$, and $\bar{p}$ is a critical point of the scalar curvature of (M, g).
We believe that the bound (5.8) is sharp in order to have smooth convergence to a *round* point (in the sense specified above); indeed, if (5.8) is violated then the sequence (Φ_k) may degenerate to a couple of bubbles, each one costing almost 4π in terms of Willmore energy.

Now let us state the main results of **(11)**. The first theorem below concerns the case of a sequence of Willmore immersions and it is a consequence of the second more general theorem about area-constrained Willmore immersions.

Theorem 37 (Laurain-Mondino 13, (11)) *Let (M, g) be a 3-dimensional Riemannian manifold and let $\Phi_k : S^2 \hookrightarrow M$ be a sequence of Willmore surfaces satisfying the energy bound (5.8) and Hausdorff converging to a point $\bar{p} \in M$.*

Then $\mathrm{Scal}(\bar{p}) = 0$ and $\nabla\mathrm{Scal}(\bar{p}) = 0$; moreover, if we rescale (M, g) around $\bar{p}$ in such a way that the rescaled immersions $\tilde{\Phi}_k$ have fixed area equal to one, then $\tilde{\Phi}_k$ converges smoothly, up to subsequences and up to reparametrizations, to a round sphere in the 3-dimensional euclidean space.

Actually we prove the following more general result about sequences of area-constrained Willmore immersions.

Theorem 38 (Laurain-Mondino 13, (11)) *Let (M, g) be a 3-dimensional Riemannian manifold and let $\Phi_k : S^2 \hookrightarrow M$ be a sequence of area-constrained Willmore surfaces satisfying the energy bound (5.8) and Hausdorff converging to a point $\bar{p} \in M$.*

Then $\nabla\mathrm{Scal}(\bar{p}) = 0$; moreover, if we rescale (M, g) around $\bar{p}$ in such a way that the rescaled immersions $\tilde{\Phi}_k$ have fixed area equal to one, then $\tilde{\Phi}_k$ converges smoothly, up to subsequences and up to reparametrizations, to a round sphere in the 3-dimensional euclidean space.

Of course Theorem 38 implies Theorem 37 except the property $\mathrm{Scal}(\bar{p}) = 0$. This fact follows by the aforementioned **[Mon1]** which for Willmore graphs over geodesic spheres, together with the smooth convergence to a round point ensured by Theorem 38.

Now let us discuss some applications to the Hawking mass m_H, defined for an immersed sphere $\Phi : S^2 \hookrightarrow (M, g)$ by

$$m_H(\Phi) = \sqrt{\frac{\mathcal{A}_g(\Phi)}{16\pi}} \left(1 - \frac{1}{4\pi} W(\Phi)\right). \tag{5.9}$$

Of course, the critical points of the Hawking mass under area constraint are exactly the area-constrained Willmore spheres (see **[LMS]** and the references therein for more material about the Hawking mass, there will be also more detail in the next section); moreover it is clear that the inequality $m_H(\Phi) \geq 0$ implies that $W(\Phi) \leq 4\pi$.

Therefore, combining this easy observations with Theorem 38, we obtain the following corollary.

Corollary 39 *Let (M, g) be a 3-dimensional Riemannian manifold and let $\Phi_k : S^2 \hookrightarrow M$ be a sequence of critical points of m_H under area constraint having non-negative Hawking mass and Hausdorff converging to a point $\bar{p} \in M$.*

Then $\nabla \mathrm{Scal}(\bar{p}) = 0$; moreover, if we rescale (M, g) around $\bar{p}$ in such a way that the rescaled immersions $\tilde{\Phi}_k$ have fixed area equal to one, then $\tilde{\Phi}_k$ converges smoothly, up to subsequences and up to reparametrization, to a round sphere in the 3-dimensional euclidean space.

Let us briefly comment on the relevance of Corollary 39 despite the triviality of its proof. Recall that, from the note of Christodoulou and Yau **[CY]**, if (M, g) has non negative scalar curvature then isoperimetric spheres (and more generally stable CMC spheres) have positive Hawking mass; on the other hand it is known (see for instance **[Druetiso]** that, if M is compact, then small isoperimetric regions converge to geodesic spheres centered at a maximum point of the scalar curvature as the enclosed volume converges to 0. Therefore a link between regions with positive Hawking mass and critical points of the scalar curvature was already present in literature, but Corollary 39 expresses this link precisely.

First of all, as it was already noticed, it is enough to prove Theorem 38 in order to get all the stated results. To prove it, we adopt the blow up technique taking inspiration from my PhD and exposed briefly in the previous section. The technical originality of the present result from the point of view of the blow-up method is that we study a *fourth order vectorial problem*.

More precisely, we consider normal coordinated centered at the limit point $\bar{p}$ and we rescale appropriately the metric g such that the rescaled surfaces have all diameter one (or, thanks to the monotonicity formula, it is equivalent to fix the area of the rescaled surfaces equal to one); notice that the rescaled ambient metrics g_k become more and more euclidean. Then, by exploiting the divergence form of the Willmore equation established in **[MR2]**, we give a decay estimate on the Lagrange multipliers as k goes to infinity. Then we prove that under the above working assumptions, the sequence (Φ_k) converges smoothly to a round sphere, up to subsequences and reparametrizations. Let us remark that in the proof we exploit in a crucial way the assumption (5.8), otherwise it may be possible for the sequence to degenerate to a couple of bubbles. Once we have smooth convergence to a round sphere ω, we study the remainder given by the difference between Φ_k and ω : we use the linearized

Willmore operator

$$-\Delta\left(\frac{\langle\Delta\rho,\omega\rangle+2\langle\nabla\omega,\nabla\rho\rangle}{|\nabla\omega|^2}\right)$$

in order to give precise asymptotics of such remainder term, once all the invariance group have been killed. Finally using these estimates and solution of the linearized operator generated by translation, we conclude the proof.

When this work was done, the energy identity for Willmore surfaces in the euclidean setting was just done, the method was not enough understood to expect applications to the Riemannian setting. But today it will be a very interesting question to make a multiple blow-up analysis in the Riemanian setting. Of course the generalization of the existing result of Zolotareva, or my uniqueness result to Willmore surfaces is a natural direction of research. For instance, Zolotareva proved that we can glue two round spheres which is of course impossible in the euclidean setting by the Hopf's theorem. Similarly, an interesting question is the following

Question 5 *Can we desingularize an inverted Enneper with a small Enneper in a Willmore surface in some Riemannian manifold ?*

This situation is also impossible in the euclidean setting by Bryant's work, since it will produce some Willmore surface with 12π of energy. This is the second basic example after gluing two spheres. In fact we cannot desingularize the inverted catenoid due to the presence of some residue.

5.3 Large CMC and Willmore surfaces in asymptotically flat manifold

As it was briefly evoked in the previous paragraph, Willmore surfaces and especially constrained area Willmore surfaces seem to play a crucial role in the context of general relativity, due to their strong link with the Hawking mass. The goal of this section is to give more detail about this relation and to expose some on going work about it.

It is a well-known fact that asymptotically flat 3-manifolds with non-negative scalar curvature are models of space-slices of space-time. There are many references on the subject, for instance [**CGP**] and references therein. Hence we will not describe in detail the physical meaning of those spaces and just focus on the link between scalar curvature, CMC surfaces and the Willmore energy.

First, let us remind some definitions. In this text we consider the case of manifolds with only one end for the sake of simplicity but most of what follows works for manifolds with multiple ends.

Definition 5 *Let (M,g) be a 3-manifold, it is said to be asymptotically flat (AF) (with one end), if there exists $\varepsilon>0$ and a compact K such that $M\setminus K$ is diffeomorphic to $\mathbb{R}^3\setminus B(0,1)$ and in those coordinates*

$$g=\delta^{ij}+O_2(|x|^{-\frac{1}{2}-\varepsilon}).$$

For such a manifold, we can define an invariant called the mass. The definition could look unnatural at first glance. But in fact it can be naturally interpreted, through Noether's theorem, as a conserved quantity in the Einstein equation.

Theorem 40 (Arnowitt, Deser, Misner, Bartnik, Chrusciel) *Let (M, g) be an asymptotically flat manifold whose scalar curvature is in L^1, then the following limit exists*

$$\lim_{R\to+\infty} \frac{1}{16\pi} \int_{S(0,R)} (g_{ij,i} - g_{ii,j})\nu^j \, d\sigma,$$

moreover it depends only on the metric. Let denote it m for mass.

Bartnik notably proved that the definition is independent of the choice of coordinates (see [**Ba**]) but more recently a definition was given which depends only on curvature, see [**Her**] and references therein. Despite this fact the dependence to the choice coordinates is a main issue in general relativity, that it is also one motivation of our work.

The more remarkable result about this invariant is probably the positive mass theorem.

Theorem 41 (Schoen-Yau 79, [SY79]) *Let (M, g) be an AF manifold with nonnegative scalar curvature. Then $m \geq 0$ with equality if and only if M is isometric to $\mathbb{R}^3$.*

This result is remarkable for many reasons. Of course it confirms the physical intuition that the total energy must be positive, but the rigidity part characterizing the flat space is just beautiful. However the more important part is probably the proof itself. Indeed Schoen and Yau use minimal surface theory and in particular their special behavior in presence of positive scalar curvature. This idea was at the origin of numerous results, which go far beyond the scope of general relativity (we refer to the following nice survey [**Cho**]).

The most symmetric manifold with positive mass, is the Schwarzschild space :

$$\left(\mathbb{R}^3 \setminus \{0\}, \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}\right).$$

It is an AF manifold with vanishing scalar curvature and mass m.

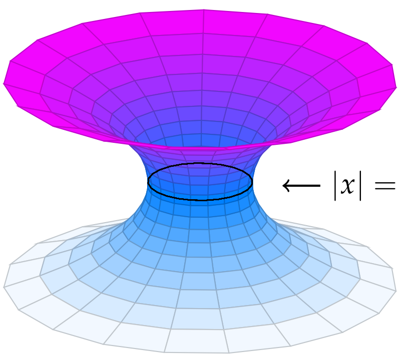

FIGURE 5.1 – The spacial geometry of the Schwarzschild metric.

Its asymptotics are particular. In fact it can be interpreted as the conformal blow-up of RP^3, or equivalently as the one obtained on S^3 with the sum of two Green functions of the conformal laplacian with antipodal singularities. Since this asymptotic plays a major role in the theory, it possesses its own denomination.

Definition 6 *A 3-manifold (M, g) is said to be Schwarzschildean (with one end), if there exists a compact K such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B(0,1)$ and in those coordinates*

$$g = \left(1 + \frac{m}{2|x|}\right)^4 \delta^{ij} + O_2(|x|^{-2}).$$

As it was already said, the mass is invariant by a change of coordinates, for instance if we consider the following version of Schwarzschild spaces : $\left(\mathbb{R}^3 \setminus \{p\}, \left(1 + \frac{m}{2|x-p|}\right)^4 \delta_{ij}\right)$, of course the mass is unchanged, but it will be interesting to define a notion of center of mass which will allow us to differentiate this space from the original. Invoking the Noether's theorem with respect to the invariance with respect to space translation, we have a similar formula than the one of the mass, but before we give a definition, we need the following one.

Definition 7 *Let (M, g) be an AF manifold, such that*

$$|g_{ij} - \delta_{ij}| + |x||\Gamma_{ij}^k| + |x|^2|Ric_{ij}| + |x|^{\frac{5}{2}}|S| \leq \frac{C}{|x|^{\frac{1}{2}+\varepsilon}}.$$

Then it satisfies the **weak Regge-Teitelboim** *condition, if*

$$|g^{odd}(x)| + |x||\Gamma^{odd}(x)| \leq \frac{C}{|x|^{1+\varepsilon}}.$$

It satisfies the **strong Regge-Teitelboim** *condition, if*

$$|g^{odd}(x)| + |x||\Gamma^{odd}(x)| + |x|^2|Ric^{odd}(x)| + |x|^{\frac{5}{2}}|S^{odd}(x)| \leq \frac{C}{|x|^{\frac{3}{2}+\varepsilon}}.$$

Theorem 42 (Beig, O'Murchadha 87, [BO87]) *Let (M, g) an AF manifold satisfying* **the strong RT condition with non vanishing mass**, *then the following limit exists*

$$\lim_{R \to +\infty} \frac{1}{16\pi m} \int_{S(0,R)} (g_{ij,i} - g_{ii,j})\nu^j x^\alpha - (g_{i\alpha}\nu^i - g_{ii}\nu^\alpha)\, d\sigma,$$

moreover it depends only on the metric. Let denote it C^α for the center of mass.

The strong RT condition was proved to be optimal by Cederbaum and Nerz in **[CN]**, where they constructed metric with divergent center of mass.

In fact the presence of mass produces some symmetry breaking, in the sense that it cannot be uniformly distributed. The goal of the rest of this section is to explain how one can read this symmetry breaking in global geometric objects and how those geometric objects can measure the mass locally.

As it was reminded in the previous sections, the behavior of CMC surfaces is strongly influenced by the scalar curvature, at least locally. So it is a reasonable idea to study the behavior of large (stable) CMC surfaces in order to detect the presence of mass. This idea is made concrete by the following theorem.

Theorem 43 (Christodoulou-Yau 88, [CY]) *Let (M, g) be a 3-manifold with none-negative scalar curvature, then the Hawking quasi-local mass*

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{4\pi} \int_\Sigma H^2 \, d\sigma\right)$$

of a closed stable CMC is non negative.

Contrary to the flat case, where stable CMC surfaces are not rigid, in the sense that a round sphere can always be translated. However, as soon as some mass is present, the situation is much more rigid. In the simplest case of Schwarzschild metric, the radial symmetry forces the geodesic spheres to be centred in order to have their mean curvature constant. In fact those spheres are the only isoperimetric one, see [**Bray**] and even the only CMC one, see [**Bre**]. The following theorem generalizes the Schwarzschild situation.

Theorem 44 (Huisken-Yau 96, Ye 97, [HuY, Ye2]) *Let M be a Schwarzschildean manifold with positive mass, then for R large enough we can perturb the sphere $S(0, R)$ into a stable CMC surface Σ_R. Those spheres form a foliation.*

The hypotheses on the decay of the metric was successively improved by Huang, Metzger and finally by Nerz who proved the existence into an AF manifold, see [**Nerz**] and reference therein.

The previous result is fundamental for two reasons. First it allows one to give a purely geometric definition of the center of mass, as follow

$$c_{HY} = \lim_{R \to 0} \frac{1}{|\Sigma_R|} \int_{\Sigma_R} x \, d\sigma,$$

where Σ_R is a leaf of the foliation produced by Huisken and Yau. The second implication is that the inverse of the mean curvature of this foliation defined an intrinsic radial coordinate, which is fundamental in a setting where most of quantities at infinity are chart dependent.

Hence a very important question has risen :

Question 6 *Are the leafs of this foliation unique ?*

We have a first answer in the case of Schwarzschildean decay with the following theorem.

Theorem 45 (Qing-Tian 07, [QT]) *Let (M, g) a Schwarzschildean manifold with positive mass. Then there exists a compact set K such that stable CMC spheres which separate the infinity from K coincide with the leafs of the CMC foliation.*

There were some slight improvements of the decay by Ma [**Ma**]. But the not-outlying condition, *i.e.* separating the compact from infinity, is necessary by the work of Brendle and Eichmair [**BE**]. However, Chodosh and Eichmair [**CE**] succeeded to get the global uniqueness adding some minimal assumptions.

However, there is no particular reason to restrict to the Schrwarzschildean case. Hence, using a similar argument than the one developed in the previous, with Metzger we get the following result, whose assumption are optimal.

Theorem 46 (In progress, Laurain-Metzger 19, (12)) *Let (M, g) be an AF manifold satisfying the* **weak RT condition***, with non vanishing mass. Then there exist a compact K and $\varepsilon_0 > 0$ and $C > 0$ such that for every CMC surfaces which separates K and infinity such that*

$$\int_\Sigma |\mathring{A}|^2 \, d\sigma \leq \varepsilon_0$$

satisfy

$$\sup_\Sigma |x| \leq C \inf_\Sigma |x|.$$

Moreover if $|\Sigma_n| \to +\infty$ then $\lim_n \dfrac{\sup_{\Sigma_n} |x|}{\inf_{\Sigma_n} |x|} \to 1$.

So any large not-outlying stable CMC sphere must be close to a leaf, and in fact equal by the implicit function theorem, which leads to the following corollary

Corollary 47 *Let (M, g) be an AF manifold satisfying the* **weak RT condition***, with non vanishing mass. Then there exists a compact K and $\varepsilon_0 > 0$ such that for every CMC surfaces which separates K and infinity such that*

$$\int_\Sigma |\mathring{A}|^2 \, d\sigma \leq \varepsilon_0$$

is a leaf of the Huisken-Yau foliation.

In this result we replace the stability assumption by the smallness of $\|\mathring{A}\|_2$, but in fact the stability automatically implies this assumption.

The proof is separated into two steps : first we prove that the surface is close to a round sphere and that it is "balanced" with respect to the compact part, that is to say the inner radius is comparable to the outer one. For this we notably use of the Schoen-Pohožev formula. Then, the second part of the proof, consists in getting some better estimates, for this we adapt the method described in the previous section.

In fact this result is a by-product of a most ambitious project, which consist in studying the behavior of large area constrained Willmore surfaces. From a physical point of view it seems even more natural to study the following variant of the isometric problem.

Question 7 *In an AF manifold what are the surfaces which maximize their Hawking mass with given area ?*

As explained in the previous section, those surfaces must be area constrained Willmore surfaces. There is even an equivalent of the Huisken-Yau foliation in this context, as it was shown by the following theorem.

Theorem 48 (Lamm-Metzger-Schulze 11, [LMS]) *Let (M, g) be a Schwarzschildean manifold with positive mass. Then there exists $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$ there exist an area-constrained Willmore surface Σ_λ satisfying $|\Sigma_\lambda| = \frac{1}{\lambda}$. Moreover there are locally unique.*

Hence the question of uniqueness naturally rises. But it seems much more difficult than the one of the CMC, not only because the Willmore equation is more technical to study, but also because the stability assumption becomes almost powerless. In fact, a classical argument in the CMC setting to get some control, works as follows : first, we make a blow-up around a point where the curvature is uncontrolled, the limit is stable minimal surface in $\mathbb{R}^3$, thanks to the result of Fischer-Colbrie and Schoen **[FCS]**, we know that it is a plane and this gives a contradiction. But in the case of Willmore surfaces, there is many stable surfaces, at least all the minimal ones, hence this important argument does not work anymore. This is also a reason why we start by studying the CMC case, not only to improve the decay, but also to get a proof without stability assumption. Our assumption on the smallness of the traceless part of the second fundamental form seems to be well adapted to Willmore.

Our first goal is to prove an equivalent of our Theorem 46 in the context of the Willmore foliations. Here is what we expect to prove,

Theorem 49 (In progress) *Let (M, g) be an AF manifold satisfying the* **weak** *RT condition,* **with non vanishing mass** *. Then there exists a compact K and $\varepsilon_0 > 0$ and $C > 0$ such that for every area constrained Willmore surfaces which separates K and infinity such that*

$$\int_\Sigma |\mathring{A}|^2 \, d\sigma \leq \varepsilon_0$$

then

$$\sup_\Sigma |x| \leq C \inf_\Sigma |x| .$$

Moreover if $|\Sigma_n| \to +\infty$ *then* $\lim_n \dfrac{\sup_{\Sigma_n} |x|}{\inf_{\Sigma_n} |x|} \to 1$.

As for the CMC case, we will have the uniqueness of the foliation as corollary. I have to say that some partial results in this direction were announced in the particular case of Schrwarzschildean manifold by Koerber **[Ko]**.

But our main objective is to revisit the notion of quasi local-mass. Since we expect that surfaces with large area which maximize their local-mass are the almost round spheres. But if we drop the not-outlying assumption, there are "exotic" competitors for the Hawking-mass. Indeed we can imagine n copies of the horizon glued by small catenoidal necks. Since the horizon is minimal, the Willmore Energy of such an object can be made as small as we want, and the Hawking mass maximal. Here is a new proposition of local-mass in order to avoid those examples. Let

$$m_\alpha(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{4\pi} \int_\Sigma (H^2 + \alpha |\mathring{A}|^2) \, d\sigma \right),$$

with $\alpha \in (0, 1/6]$. The Choice of $1/6$ is not arbitrary. If one go through the proof of Theorem 43, one can easily check that the conclusion is true for m_α as soon as $\alpha < \frac{1}{6}$. Hence this new local-mass shares many common property with the Hawking mass and moreover the presence of the L^2 norm of the traceless part of the second fundamental form rules out the formation of neck in the maximizing sequences. Finally, our assumption on the smallness of the $\|\mathring{A}\|_2$ is almost automatically satisfied. So we expect a global uniqueness result in the spirit of the ones obtained for CMC surfaces by Chodosh and Eichmair.

Another direction of research we have with Metzger consists in detecting the angular momentum using the CMC foliation. The angular momentum, is the conserved quantity of the Einstein equation given by the invariance by rotation. The idea consists in producing a geometric definition which relies on the behavior of the CMC foliation. Indeed we expect that the rotation of the space flattens the CMC foliation where it meets the axis of rotation. The problem is that this deformation appears at the next order with respect to one produced by the mass and the position of the center of mass. So we need to subtract carefully the effect of the mass before trying to detect this invariant. To conclude, we would like to underline that this intuition is nothing else than the equivalence principle from a mathematical point of view : gravity is replaced by deformation of space and scalar curvature can be detected in deformations of CMC or Willmore surfaces.

Chapitre 6

Energy convexity and applications

This chapter is based on the following two articles : **(9)** and **(10)**.

In this chapter we will investigate the convexity of some natural functionals such as the Dirichlet energy or biharmonic energies. Indeed, let (M, g) be a manifold with non-positive curvature, it is a well-know fact that the square of the distance function, $d : M \times M \to \mathbb{R}$, is convex. In particular, we easily get that any geodesic joining two points is unique. Those facts, still in the context of non-positive curvature, easily generalizes to harmonic maps (see section 8 of [**Jost**]).

As soon as the curvature is positive, things become more complicated. For the same reason that there is no more one unique geodesic joining two points, harmonic maps with the same boundary data are not unique anymore when the curvature is positive. In particular, it becomes impossible to speak about "the harmonic map" with a given boundary and we need to pay much more attention when we want to do some replacement procedure.

The goal of this chapter is to explain how we can overcome this difficulty.

6.1 Energy convexity for harmonic maps a new proof

Using the Bochner method, we can easily see that in a Riemannian manifold with non-positive curvature, the second variation of an harmonic maps is non-negative, in such a way that the Dirichlet energy is locally convex and more generally we can deduce some uniqueness results with fixed boundary data. But considering the simple case of the round sphere and a circle as boundary data, as in the following picture, we can easily observe that uniqueness is lost when curvature is positive.

The red solution is stable and the black one is clearly unstable. We sometimes speak about the small and the big solution, the following theorem proved that if a solution is not too big then the Dirichlet energy is locally convex around it. This ensures local uniqueness despite the general non-uniqueness.

Theorem 50 (Colding-Minicozzi 08, [CM08]) *Let (M, g) a compact Riemannian manifold, there exists $\varepsilon_0 > 0$ such that if $u, v \in W^{1,2}(\mathbb{D}, M)$ with $u_{|\partial\mathbb{D}} = v_{\partial\mathbb{D}}$ and u is a harmonic map with $\|\nabla u\|_2^2 \leq \varepsilon_0$ then*

$$\frac{1}{2}\int_{\mathbb{D}} |\nabla(v-u)|^2\, dx \leq \int_{\mathbb{D}} |\nabla v|^2\, dx - \int_{\mathbb{D}} |\nabla u|^2\, dx.$$

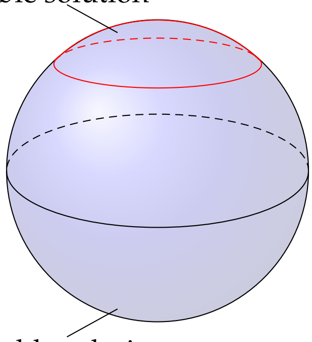

FIGURE 6.1 – None-uniqueness of harmonic maps

Which gives as an immediate corollary.

Corollary 51 *Let (M,g) a compact Riemannian manifold, there exists $\varepsilon_0 > 0$ such that if $u, v \in W^{1,2}(\mathbb{D}, M)$ are harmonic maps with energy smaller than ε_0 and with $u_{|\partial\mathbb{D}} = v_{\partial\mathbb{D}}$ then $u \equiv v$.*

Let us sketch briefly the proof of Theorem 50 in order to understand how to extract the main ideas to generalize it.

Sketch of the proof of Theorem 50 :

Once more, using Nash embedding theorem, we can assume that (M,g) is isometrically embedded into some $\mathbb{R}^N$. Then we get

$$\begin{aligned}\int_{\mathbb{D}} |\nabla v|^2\,dx - \int_{\mathbb{D}} |\nabla u|^2\,dx - \int_{\mathbb{D}} |\nabla (v-u)|^2\,dx &= 2\int_{\mathbb{D}} \langle \nabla(v-u), \nabla u\rangle\,dx \\ &= -2\int_{\mathbb{D}} \langle v-u, \Delta u\rangle\,dx \\ &\geq -2C\int_{\mathbb{D}} |v-u|^2|\Delta u|\,dx \\ &\geq -2C\int_{\mathbb{D}} |v-u|^2|\nabla u|^2\,dx,\end{aligned} \tag{6.1}$$

here we used the fact that since u is harmonic we have $\Delta u \in (T_u M)^\perp$ and that $(v-u)^\perp \leq C|v-u|^2$ in a compact submanifold. Then, to get the result, it suffices to prove that

$$\int_{\mathbb{D}} |v-u|^2 |\nabla u|^2\,dx \leq C\varepsilon_0 \int_{\mathbb{D}} |\nabla(v-u)|^2\,dx, \tag{6.2}$$

which is a consequence of the following lemma

Lemma 52 *There exists $\varepsilon_0 > 0$ such that if $u \in W^{1,2}(\mathbb{D}, \mathbb{R}^N)$ is a harmonic map with $\|\nabla u\|_2^2 \leq \varepsilon_0$, then for all $h \in W_0^{1,2}(\mathbb{D}, \mathbb{R}^N)$, we have*

$$\left(\int_{\mathbb{D}} h^2 |\nabla u|^2\,dx\right) \leq C\left(\int_{\mathbb{D}} |\nabla h|^2\,dx\right)\left(\int_{\mathbb{D}} |\nabla u|^2\,dx\right)$$

for some $C > 0$.

The proof relies on the existence of a Coulomb frame (e_i), as originally derived by Hélein, such that setting

$$\alpha_i = \left\langle \frac{\partial u}{\partial z}, e_i \right\rangle,$$

there exists $\beta : \mathbb{D} \to Gl_n(\mathbb{C})$, with a uniform control on its norm, such that

$$\frac{\partial}{\partial \bar{z}}(\beta^{-1}\alpha) = 0.$$

Then we can derive lemma 52 from the following general lemma

Lemma 53 (Proposition C.1 of [CM08]) *If f is holomorphic on $\mathbb{D}$ and $h \in W_0^{1,2}(\mathbb{D})$ then*

$$\left(\int_{\mathbb{D}} h^2|f|^2\,dx\right) \leq 8\left(\int_{\mathbb{D}} |\nabla h|^2\,dx\right)\left(\int_{\mathbb{D}} |f|^2\,dx\right).$$

□

Despite the elegance of the proof, the fact that it reduces our problem to some complex analysis is not totally satisfactory since there is few hope to generalize it to dimension 4 for biharmonic maps, for instance. More recently, Lamm and Lin proposed a new proof based on Rivière's work (see **[LL]**). As seen in chapter 3, Rivière found some conservation laws to make a Jacobian structure appear into conformally invariant equations. Moreover, Coifman, Lions, Meyer and Semmes, proved in **[CLMS]**, that the Jacobian of a $W^{1,2}$ map is not only in L^1 but in $\mathcal{H}^1$, the Hardy space. This property plays a fundamental role in the compactness by compensation phenomena, since as said by Hélein in section 3.2 of **[He]**,

> *" A second way is to think of Hardy space as the biggest subspace of L^1 for which we can use the recipes valid in L^p for $1 < p < \infty$, but which are no longer valid for L^1".*

Especially some Calderon-Zygmund theory is available for Hardy space.

In fact Lamm and Lin proved something a bit stronger in their paper. They proved that not only the right hand side of the harmonic map equation is in Hardy but that for an harmonic map $|\nabla u|^2 \in \mathcal{H}^1$. Then they derived the following lemma.

Lemma 54 *There exists $\varepsilon_0 > 0$ such that if $u \in W^{1,2}(\mathbb{D}, M)$ is harmonic with $\|\nabla u\|_2^2 \leq \varepsilon_0$, then there exists $\phi \in W_0^{1,2} \cap L^\infty$ such that*

$$\Delta\phi = |\nabla u|^2$$

and $C > 0$, depending only on M, such that

$$\|\phi\|_\infty + \|\nabla\phi\|_2 \leq C\|\nabla u\|_2^2.$$

Then, by integration by parts, we easily get the following desired estimate,

$$\int_{\mathbb{D}} |v-u|^2|\nabla u|^2\,dx = \int_{\mathbb{D}} |v-u|^2\Delta\phi\,dx \leq C\varepsilon_0 \int_{\mathbb{D}} |\nabla(v-u)|^2\,dx.$$

Hence this new proof, which essentially relies on a Hardy space estimate which is done thanks to Rivière's work has good chance to be generalized to many comparable situations. Unfortunately, the Hardy space is rather difficult to deal with, especially due to its technical definition. But going through the proof of Lemma 54, I realized that the Hardy estimate is not absolutely needed, but its by-products only. The following lemma sums up this idea.

Lemma 55 *If $f \in L^1(\mathbb{D})$ and there exists $C_0 > 0$ such that*

$$|f(x)| \leq C_0 \frac{\|f\|_1}{(1-|x|)^2}. \tag{6.3}$$

Then there exists $\phi \in W_0^{1,2} \cap L^\infty$ such that

$$\Delta\phi = f$$

and $C > 0$, such that

$$\|\phi\|_\infty + \|\nabla\phi\|_2 \leq CC_0\|f\|_1.$$

The property (6.3) can seem strange at first glance but it is almost equivalent to an ε-regularity when $f = |\nabla u|^2$. Indeed, if we apply ε-regularity at x on a ball of radius $\frac{1-|x|}{2}$ we get

$$|\nabla u(x)| \leq \frac{C}{(1-|x|)}\|\nabla u\|_2.$$

In some sense, we prove that, despite the fact that $f \in L^1$ does not imply (6.3) and reciprocally, **those two properties together are equivalent to some Hardy estimate**. Moreover they are much more easy to check, especially for any solution of a conformally invariant problem. This idea has not been published since we even find an easier proof of the convexity, but I think that it could be very useful each time we have to deal with the Hardy space. Let us give a brief sketch of the proof of the previous lemma.

Idea of the proof of Lemma 6.3 :

It is classical that, by integration by part, the L^∞-estimate implies the L^2-estimate. Hence we write that

$$\phi(x) = \int_{\mathbb{D}} G(x,y) f(y)\, dy,$$

where G is the Green function of the Laplacian. Then, we remark that the singularity of the Green function can be decomposed as a sum of bump functions with dyadic support, namely

$$l_x(y) - C \leq G(x,y) \leq l_x(y) + C,$$

with

$$l_x(y) = \sum_{j=1}^{\infty} \Theta(2^j(1-|x|)^{-1}(x-y)),$$

where Θ is a smooth cut-off function, such as $\mathrm{supp}(\Theta) \subset B_{\frac{1}{8}}$, of the following form

Then

$$|\phi(x)| \leq C\|f\|_1 + \sum_{j=1}^{\infty} \int_{\mathbb{D}} \Theta(2^j(1-|x|)^{-1}(x-y)) f(y)\, dy$$

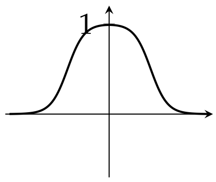

FIGURE 6.2 – Cut-off function

Denoting by I_j each integral of the right hand side we easily see that

$$\begin{aligned} I_j &= \int_{\mathbb{D}} \Theta(2^j(1-|x|)^{-1}(x-y))f(y)\,dy \\ &\leq \int_{B(x,2^{-j-3}(1-|x|))} f(y)\,dy \\ &\leq |B(x,2^{-j-3}(1-|x|))|\frac{2C\|f\|_1}{(1-|x|)^2} \\ &\leq 2^{-2j-5}\pi C\|f\|_1 \end{aligned} \tag{6.4}$$

Finally we just have to sum this inequality to get the result. □

Despite the simplicity of this argument, it could not be generalized to all situations. Since, in the harmonic maps case, the right hand side is in fact very specific. Indeed it behaves more or less like $|\nabla u|^2$, that is to say its L^1-norm is trivially bounded by the energy. But as we will see later with biharmonic maps, the right hand side is not automatically in L^1. Finally thanks to the Hardy inequality, see section 1.3.1 of [**Maz**], see also [**MMP**], we can derive **a one line proof** of the desired estimate. Let us remind the Hardy inequality.

Theorem 56 *Let $h \in W^{1,2}_0(\mathbb{D})$ then*

$$\frac{1}{4}\int_{\mathbb{D}} \frac{h^2}{(1-|x|)^2}dx \leq \int_{\mathbb{D}} |\nabla h|^2\,dx\,. \tag{6.5}$$

Then we have just to apply consecutively the ε-regularity and the previous Hardy inequality to prove (6.2), as follows

$$\int_{\mathbb{D}} |v-u|^2|\nabla u|^2\,dx \leq C\varepsilon_0 \int_{\mathbb{D}} \frac{|v-u|^2}{(1-|x|)^2}\,dx \leq \frac{1}{2}\int_{\mathbb{D}} |\nabla(v-u)|^2\,dx,$$

choosing ε_0 small enough.

Before going through some applications of this idea in the next sections, we would like to point out some facts on the Hardy inequality. Indeed the constant $\frac{1}{4}$ is not surprising at all in (6.5), since as it was already remarked by several authors and very nicely developed by Devyver (see [**Dev**] and reference therein) the best constant in the Hardy inequality is nothing else than the bottom of the spectrum of an elliptic operator on a domain with a weighted metric. In our present case $\frac{1}{4}$ is the bottom of the spectrum of the laplacian on the hyperbolic disc. So the Hardy inequality can be seen either as a weighted Poincaré inequality or a Poincaré inequality with the adapted metric. The fact that the Möbius group is the group of invariance of the Dirichlet energy from one hand but also the isometry group of the hyperbolic disk from the other hand reinforces this idea.

6.2 Energy convexity of free boundary harmonic maps and width of manifold with boundary

In this section we are interested in the seminal approach of the existence theory of minimal surfaces, the one used by Douglas and Radó to solve the Plateau problem in the 1930's, namely the variational approach. In this issue, the natural continuation to look for critical points is a min-max, a setting which finds its foundations in the work of Palais [**Palais**]. A min-max version of this approach was performed in the very nice paper by Colding-Minicozzi : [**CM08**], which especially permits to define the *width* of a 3-sphere. Inspired by the work by Douglas and Radó they replaced the area by the Dirichlet energy functional, and inspired by the Birkhoff's so-called curve-shortening procedure, they gave a strong harmonic replacement procedure, based on the local convexity of the energy described in the previous section. Following the work by Colding-Minicozzi, we extend this result to free-boundary minimal surfaces. Those surfaces can be seen as a generalization of closed geodesics, in the sense that the boundary of the disk is a 1/2-harmonic map which is the 1-dimensional equivalent of the harmonic map (see section 3.6 for more detail about the non-local approach).

The main goal of **(9)** is to perform a free-boundary harmonic replacement procedure in order to produce a free boundary minimal surface whose area achieves the min-max value over all disk sweepouts of a manifold whose boundary lies in a submanifold.

Let N be a n-dimensional closed manifold and $M \subset N$ a connected m-dimensional compact submanifold. We let

$$\mathcal{A} = \{u \in W^{1,2}(\mathbb{D}, N) \cap \mathcal{C}^0(\overline{\mathbb{D}}, N) \mid u(\partial\mathbb{D}) \subset M\}$$

be the set of admissible parametrized disks in N whose boundaries lie in M, endowed with the $\|\,.\,\|_{L^\infty} + \|\nabla\,.\,\|_{L^2}$-norm. We fix a parameter $t_0 \in \partial\mathbb{B}^{k-2}$ and an arbitrary point $m_0 \in M$. We define a sweepout as a map $\sigma : \mathbb{B}^{k-2} \to \mathcal{A}$ on a $k-2$-ball which is the set of parameters with $k \geq 2$ satisfying

— $\forall t \in \partial\mathbb{B}^{k-2}$, σ_t is a constant function in M.

— $t \mapsto \sigma_t$ is continuous in $\mathcal{A}$.

— $\sigma_{t_0} = m_0$ [1].

Let ω be a homotopy class of sweepouts, we can set a topological invariant, called the *width* as

$$W(N, M, \omega) = \inf_{\sigma \in \omega} \max_{t \in \mathbb{B}^{k-2}} Area(\sigma_t). \tag{6.6}$$

Notice that there is a non-trivial homotopy class of sweepouts (or there is a sweepout non-homotopic to a constant one) if and only if $\pi_k(N, M) \neq \{0\}$ (see section 1.3 of [**F00**]). In particular in those cases, $W(N, M, \omega) > 0$ and our main theorem applies. Our setting is very general since it contains the classical Plateau problem when M is a closed curve and $k = 2$. And it of course contains also some real min-max problems, for instance, if M is the boundary of a strictly-convex domain in $\mathbb{R}^3$, the level set of the height function generates a non-trivial homotopy class if $\pi_3(N, M) \neq \{0\}$.

1. the condition that $\sigma_{t_0} = m_0$ can be removed when M is simply connected. Else, it is necessary to get that $\pi_{k-2}(\mathcal{A}) \cong \pi_k(N, M)$.

The definition of the classical *width* goes back to Birkhoff **[Bir]** for the problem of geodesics : it is the smallest length we need for some circle to pull-over a compact manifold. In our context, we define the smallest area we need to cross over a compact manifold with an interface which has the topology of a disk whose boundary slides on this compact manifold.

Like for the problem of geodesics or the Douglas-Radó approach, it is more convenient to use the energy as a functional instead of the area. We know that for $u \in \mathcal{A}$, $\text{Area}(u) \leq E(u) := \frac{1}{2}\int_{\mathbb{D}} |\nabla u|^2$ with equality if and only if $u : \mathbb{D} \to N$ is conformal. Moreover, on a disk, for any immersion $u \in \mathcal{A}$, we can change the parametrization so that $u \circ \phi \in \mathcal{A}$ is an "almost-conformal" map, and we can do it continuously along any sweepout without changing the homotopy class (see the appendix D in **[CM08]**). Therefore, we have

$$W(N, M, \omega) = \inf_{\sigma \in \omega} \max_{t \in \mathbb{B}^{k-2}} \frac{1}{2} \int_{\mathbb{D}} |\nabla \sigma_t|^2. \tag{6.7}$$

The critical maps $u : \mathbb{D} \to N$ of the energy with the constraint that $u(\partial\mathbb{D}) \subset M$ are free-boundary harmonic maps, that is harmonic maps such that $\partial_\nu u \in (T_u M)^\perp$ on $\partial\mathbb{D}$. Notice that looking at the Hopf differential, a harmonic map with free boundary on the disk is automatically conformal and then minimal. This means that the set of critical points is the same considering either the energy or the area. Here is our main theorem.

Theorem 57 (Laurain-Petrides 18, (9)) *Let N be a n-dimensional closed regular manifold and M be a m-dimensional compact submanifold, let ω a homotopy class of sweepouts such that $W(N, M, \omega) > 0$. Then, there is a minimizing sequence of sweepouts $\sigma^n \in \omega$ such that for any sequence of parameters $t_n \in \mathbb{B}^{k-2}$ satisfying*

$$Area(u_n) \to W(N, M, \omega) \text{ as } n \to +\infty$$

with $u_n = \sigma^n_{t_n}$, then, up to a subsequence, there exists $r \geq 0$ (possibly branched) minimal disks with free boundary in M, $\theta_i : \mathbb{D} \to N$ and $s \geq 0$ (possibly branched) minimal spheres $\omega_j : S^2 \to N$ such that

$$u_j \to \sum_{i=1}^{r} \theta_i + \sum_{j=1}^{s} \omega_j$$

in the sense of the $W^{1,2}$-bubble convergence[2]. *Moreover, they achieve the width, i.e.*

$$W(N, M, \omega) = \sum_{i=1}^{r} \text{Area}(\theta_i) + \sum_{j=1}^{s} \text{Area}(\omega_j).$$

Remark : Bubble convergence implies varifold convergence, as it was proved in A.3 of **[CM08]**.

This gives an existence theorem as soon as $\pi_k(N, M) \neq 0$, either there is a minimal disk with free boundary, or a minimal sphere. This existence part was already obtained by Fraser **[F00]**. Notice that our limiting surfaces are not *a priori* embedded and can even possess some isolated branched points. Thanks to a min-max method

2. See definition 4 for a precise definition.

using geometric measure theory tools, Li [**Li**] and Li-Zhou [**LZ**] proved the existence of properly embedded minimal disks in the case when M is the boundary of a domain in $\mathbb{R}^n$, without convexity assumption on M. However, their disk does not a priori achieve the *width*.

Our conclusion is optimal in the general case, since examples where the limiting surface is a union of disconnected spheres or disks have to occur. We cannot either expect $\mathcal{C}^0$-bubble convergence, and of course, the limit is not *a priori* in the same homotopy class, even in the minimization case. For instance, consider a manifold which contains a minimal sphere which encloses a singularity and which is asymptotically flat, as the Schwarzschild space. If we consider M a curve being far from the minimal horizon and we try to minimize the area of disk in the homotopy class that encloses the singularity then we will blow at least one minimal horizon and possibly a minimal disk that does not enclose the horizon, as shown by the following picture

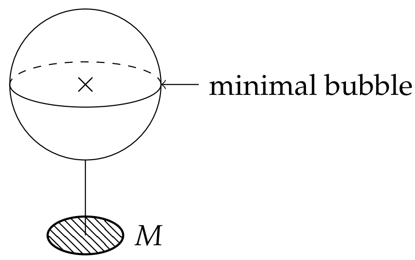

FIGURE 6.3 – minimal bubble in minimization process

It is a very interesting question to know if the *width* can be achieved by a single surface. We can easily exclude interior blow-up points assuming there is no minimal sphere in N which is the case as soon as the curvature of N is non-positive for instance, see corollary 8.6.3 of [**Jost**]. In the general case we expect some bubbling phenomena to occur at the boundary. Nevertheless we expect the situation to be much more rigid when M is the boundary of a two-convex domain since in this case the minimal disk must stay in the interior.

Definition 8 (Definition 1.2 of [F02]) *A hypersurface M in a Riemannian manifold N is two-convex if the sum of any pair of principal curvatures of M with respect to the inward pointing unit normal is positive. A two-convex domain is a domain with smooth two-convex boundary.*

Since, the boundary of a two-convex domain is a barrier for minimal disk, see proof of theorem 2.1 in [**F02**], we have the following immediate corollary

Corollary 58 *Let N a two-convex domain in $\mathbb{R}^p$ or in manifold with non-positive curvature, let ω a homotopy class of sweepouts such that $W(N, \partial N, \omega) > 0$ then there is a minimizing sequence of sweepouts $\sigma^n \in \omega$ such that for any sequence of parameters $t_n \in \mathbb{B}^{k-2}$ satisfying*

$$Area(u_n) \to W(N, \partial N, \omega) \text{ as } n \to +\infty$$

with $u_n = \sigma^n_{t_n}$, then, up to a subsequence, there exist $r \geq 1$ minimal disks with free boundary in ∂N, $\theta_i : \mathbb{D} \to N$ such that

$$u_n \to \sum_{i=1}^{r} \theta_i$$

in the sense of the $W^{1,2}$-bubble convergence. Moreover, they achieve the width, i.e.

$$W(N, \partial N, \omega) = \sum_{i=1}^{r} \text{Area}(\theta_i).$$

Finally, inspired by the result of Fraser, Theorem 1 of **[F00]** (see also **[Ri18]**) we should be able to prove in an incoming work that the sum of the indices of the minimal disks is at most $k-2$. Hence, using the fact that the index of a minimal disk in a convex domain of $\mathbb{R}^p$ is at least $[(p-2)/2]$, see Theorem 2.5 of **[F00]**, we will have an upper bound on r in the convex case, especially that $r=1$ for a convex domain in $\mathbb{R}^3$ and get a universal bound on r with respect to the topology in $\mathbb{R}^p$ with $p \geq 3$. See also **[ACS]** and **[ChMa1, ChMa2]**, for a relation between the index and the topology of free boundary minimal surfaces.

For a general domain, when $r > 1$ it will be a very interesting question to localize the blow-up point and to describe the possible configuration of free boundary minimal disks that may occur.

Our study can be viewed as the free-boundary counterpart of the classical existence results of harmonic maps on a 2-sphere into closed manifolds initiated by Sacks-Uhlenbeck in the minimization case **[SU]** and Micallef-Moore for the min-max **[MM]**. Even in the closed case, the main problem to apply the classical methods for min-max is that we need some Palais-Smale assumption for the energy functional. We need that a sequence of maps $u_n \in W^{1,2}(\mathbb{S}^2, N)$ satisfying

$$\Delta u_n = A(u_n)(\nabla u_n, \nabla u_n) + f_n \tag{6.8}$$

converges in $W^{1,2}$ up to a subsequence, where A is the second fundamental form given by the embedding $N \subset \mathbb{R}^p$ and $f_n \to 0$ as $n \to +\infty$ in $W^{-1,2}$. Of course, when $f_n = 0$, it is the harmonic map equation. It is well known that even for $f_n = 0$, we cannot expect $W^{1,2}$-convergence. The optimal result was given by Parker, see **[Pa]**, with a $W^{1,2} \cap C^0$-bubble convergence. However, in this paper, Parker also gave an example of u_n satisfying (6.8) which cannot even converge in the $W^{1,2}$-bubble convergence sense.

Notice that we have more information than (6.8) since the problem comes from a min-max, and even if we renounce to prove that any minimizing sequence for the min-max converges we can select a suitable one by a regularization process. There are two classical strategies.

The first one, the so-called viscosity method is to change the functional depending on a smoothing parameter α so that it satisfies the Palais-Smale assumption for the regularized functional and prove the convergence of the critical min-max solutions associated to α as $\alpha \to \alpha_0$. In this case, we have of course a sequence of solutions of (6.8), but they are also critical points of the regularized functional. Therefore we add another structure on the sequence of equations and we can expect convergence. This approach was introduced by Sacks-Uhlenbeck using the α-energy

$$E_\alpha(u) = \frac{1}{2}\left(\int_{\mathbb{S}^2} (1 + |\nabla u|^2)^\alpha - \int_{\mathbb{S}^2} 1\right)$$

where $E_\alpha \to E$ as $\alpha \to 1$. It suffices to study a sequence of critical points of E_α, satisfying the "α-harmonic map equation" and hope for some convergence as $\alpha \to 1$. The main step they proved is an ε-regularity independent from α, leading to the celebrated existence theorem. Fraser used the same method to prove the existence part in the free-boundary case, adding an important work for free-boundary regularity. However, we need another step in order to prove bubble convergence and energy identities : the "no-neck energy" lemma. Unfortunately, such a lemma is not true a priori for a sequence of α-harmonic maps, this was proved by Li-Wang [**LW**]. Again, of course, we have more information than the α-harmonic map equation : the sequence comes from min-max solutions, and thanks to the monotonicity trick by Struwe [**Struwe**], up to a subsequence, one can add an entropic condition that Lamm used [**Lamm10**] to prove the $W^{1,2}$-bubble convergence. The viscosity method was also performed by Rivière in the extrinsic case, adding to the area functional [**Riv17**] or Willmore functional [**Riv15**] adapted smoothers depending on a L^p-norm of the second fundamental form.

The second strategy is to start from a minimizing sequence and to replace it with a suitable procedure by a new competitor which satisfies conditions leading to compactness. This idea goes back to Birkhoff in the case of closed geodesics, with the famous curve-shortening process : choosing a set of points sufficiently close, we replace each portion between two successive points by the unique geodesic which joins these points. Then, we can use the compactness of the space of sweepouts of piecewise geodesic curves. We use this strategy to prove Theorem 57, based on the modern adaptation by Colding and Minicozzi [**CM08**], and their proof of $W^{1,2}$-bubble convergence for the min-max problem for 2-spheres in a compact manifold diffeomorphic to $\mathbb{S}^3$. Generalizations for tori and higher genus surfaces are performed by Zhou in [**Z10**] and [**Z17**] where harmonic maps are not automatically conformal. In the next section, we will give the main steps of proof of Theorem 57, which handles this strategy for free boundary issues.

In order to perform the replacement procedure, one needs to get at least the uniqueness of harmonic maps with small energy with respect to the Dirichlet boundary data. Even though it is a direct consequence of maximum principle when the target manifold is $\mathbb{R}^p$ it is much harder in the general case. Colding and Minicozzi proved in fact much more, since they proved the energy convexity of the energy functional using some property of holomorphic functions, see section 6.1. In fact, the crucial point is the compactness compensation phenomena which appears in all conformally invariant problems, see Section 6.1. In **(9)** we prove the following free boundary version of Colding-Minicozzi energy convexity.

Theorem 59 *Let N be a compact submanifold of $\mathbb{R}^n$ and M a closed submanifold of N. Then, there exists a constant $\varepsilon_0 > 0$ such that if $u, v \in W^{1,2}(\mathbb{D}_+, N)$ with $u|_A = v|_A$, for almost every $x \in I$, $v(x) \in M$ and $u(x) \in M$, $E(u) \leq \varepsilon_0$ and u is weakly harmonic meeting M orthogonally along I, then we have the energy convexity*

$$\frac{1}{2}\int_{\mathbb{D}_+} |\nabla(v-u)|^2 \leq \int_{\mathbb{D}_+} |\nabla v|^2 - \int_{\mathbb{D}_+} |\nabla u|^2 \; . \tag{6.9}$$

Where $\mathbb{D}_+ = \mathbb{D} \cap \{(x_1, x_2) \,|\, x_2 > 0\}$ and $\partial\mathbb{D}_+ = A \cup I$ with $I = (-1, 1) \times \{0\}$ and $A = \partial\mathbb{D}_+ \setminus I$.

This kind of energy convexity is also of first importance to get some strong convergence of the flow associated to some conformally invariant problem such as harmonic maps or bi-harmonic maps, see [**Lin**] and the next section. Moreover, using the flow as another smoother for sweepouts could lead to some compactness, as in the work of Fraser-Schoen [**FS2**]. Notice that energy identities for harmonic flow have already been performed, see for instance [**DT**].

In the rest of the section we describe the strategy of the proof of Theorem 57. As we already mentioned, the proof follows the strategy of Colding-Minicozzi for the sphere case, we will mainly focus on the new difficulties given by the presence of a boundary. Especially we have to develop new tools since the classical arguments which permit to get estimates by reducing the size of the set does not work at the boundary.

By Nash embedding theorem, we may assume in the rest of the paper that N is isometrically embedded in some $\mathbb{R}^p$.

Step 0 : $W = W_E$

Setting

$$W_E(N, M, \omega) = \inf_{\sigma\in\omega} \max_{t\in\mathbb{B}^{k-2}} \frac{1}{2}\int_{\mathbb{D}} |\nabla\sigma_t|^2,$$

we have $W_E = W$.

This was the seminal idea of Douglas-Radó to work on the energy rather than the Area. The main idea is that any $W^{1,2}$-map can be reparametrized as a quasi-conformal map for which the Area and the Energy are as close as desired. Colding and Minicozzi carefully improved this idea to the sweepout setting. The main point is that the reparametrization has to be continuous with respect to the parameters of the sweepout. Since the sphere case and the disk case are similar, they both possess only one conformal class, we just remind the main three steps of the Appendix D of [**CM08**].

— It is clear that $W \leq W_E$. Then, let us consider σ such that $\max_{t\in\mathbb{B}^{k-2}} Area(\sigma_t) \leq W + \frac{\varepsilon}{2}$, for some $\varepsilon > 0$.

— Then, in lemma D.1 of [**CM08**], improving the density argument of Schoen-Uhlenbeck, see last proposition of [**ScU**], Colding-Minicozzi explained how to regularize the sweepout σ to some $\tilde{\sigma} \in C^0(\mathbb{B}^k, C^2(\mathbb{D}, N))$, verifying $\max_{t\in\mathbb{B}^{k-2}} \mathrm{Area}(\tilde{\sigma}_t) \leq W + \varepsilon$. We have to notice that in our case the boundary of the mollified sweepout is *a priori* not in M. In order to solve this issue, it suffices to consider the family of C^0-curves $c_t = \sigma_{t|\partial\mathbb{D}}$, and to regularize it to some $\tilde{c}_t$ which still take values into M. Then we consider $\tilde{v}_t = \pi_N(h_t)$ where h_t is the harmonic extension of $\tilde{c}_t$ into $\mathbb{R}^p$ and π_M the projection onto N. This $\tilde{v}_t$ is a smooth map close to $\tilde{\sigma}_t$ in some neighborhood of $\partial\mathbb{D}$. Finally it suffices to interpolate $\tilde{\sigma}_t$ and $\tilde{v}_t$ in this neighborhood to get a C^2 sweepout, still denoted by $\tilde{\sigma}_t$, which is $W^{1,2} \cap C^0$ close to σ_t and which sends $\partial\mathbb{D}$ to M.

— Considering the pullback metric $\tilde{\sigma}_t^*(\xi)$ on the disk, it can be degenerated, hence one considers $g_{\delta,t} = \tilde{\sigma}_t^*(\xi) + \delta|dz|^2$ with $\delta > 0$. Then thanks to the Riemann mapping theorem for variable metric of Ahlfors-Bers **[AB]**, see also section 3.2 of Jost **[Jo]**, one finds a unique conformal diffeomorphism $h_{\delta,t}$ which fixes three points on the boundary and pulls back $g_{\delta,t}$ to $|dz|^2$, with some $W^{1,2} \cap C^0$ control on the diffeomorphism.

— Finally using this control and choosing $\delta > 0$ small enough, Colding-Minicozzi proved that the Energy of $\tilde{\sigma}_t \circ h_{\delta,t}$ can be made as close as we want to the *Area* of σ_t.

Step 1 : ε-regularity for a sequence of harmonic maps with free boundary of uniformly bounded energy.

An ε-regularity convergence for a sequence of critical points has to be true for Theorem 57. This result was proved by Laurain-Petrides **(6)**, we refer to section 3.6 for more details.

Step 2 : Convexity for free boundary energy.

This is a key step of the paper of Colding and Minicozzi in order to define a replacement procedure. In the classical problem of geodesics by Birkhoff, we replace portions of curves by a geodesic which joins their ends, but we need the uniqueness of geodesics : the points have to be below the injectivity radius. We refer to theorem 59 for the full statement. We can deduce the following uniqueness result as an immediate corollary.

Corollary 60 *Let N be a compact submanifold of $\mathbb{R}^p$ and M a closed submanifold of N. Then, there exists a constant $\varepsilon_0 > 0$ such that if $u, v \in W^{1,2}(\mathbb{D}_+, N)$ are weakly harmonic meeting M orthogonally along I, with $u|_A = v|_A$, $E(u) \leq \varepsilon_0$ and $E(v) \leq \varepsilon_0$ then $u \equiv v$.*

As it was explained in Section 6.1, it relies only on the ε-regularity (we refer to this section for more details).

Step 3 : The harmonic replacement procedure and decreasing energy map

In the proof of Theorem 57, we first take a minimizing sequence of sweepouts. Of course, we extract a classical Palais-Smale sequence u_n (satisfying (6.8) in the interior) but as already said, we cannot conclude for $W^{1,2}$-bubble convergence. We aim at replacing this sequence by another one which will satisfy stronger Palais-Smale-like properties used in *STEP 4* to prove convergence.

Let us sketch this procedure in the particular case of minimization (the set of parameters is trivial $\mathbb{B}^{k-2} = \{0\}$).

Let $u \in \mathcal{A}$. Let $\mathcal{E}$ the set of finite families of disjoint element of $\mathfrak{B}$ and $\mathfrak{H}\mathfrak{B}$ where

$$\mathfrak{B} = \{\overline{B(a,r)} \text{ s.t. } B(a,r) \subset \mathbb{D}\}$$

and

$$\mathfrak{H}\mathfrak{B} = \{\overline{B(a,r) \cap \mathbb{D}} \text{ s.t. } a \in \partial\mathbb{D} \text{ and } \partial B(a,r) \text{ intersect orthogonally}\}.$$

For $\alpha \in (0,1]$, and $\mathcal{B} \in \mathcal{E}$ we denote by $\alpha\mathcal{B}$ the collection of concentric disks to those of $\mathcal{B}$ with radius dilated by α and half disks which meet the boundary orthogonally, with center collinear to those of $\mathcal{B}$ and with radius dilated by α.

For and $\mathcal{B} \in \mathcal{E}$ such that the energy of u on $\bigcup_{B\in\mathcal{B}} B$ is less than the ε_0 of Theorem 59, then we denote by $H(u,\mathcal{B}) : \mathbb{D} \to N$ the map that coincides

— on $\mathbb{D} \setminus \bigcup_{B\in\mathcal{B}} B$ with u

— on $\bigcup_{B\in\mathcal{B}} B$ with the (unique) energy minimizing map from $\bigcup_{B\in\mathcal{B}} B$ to N that agrees with u on $\bigcup_{B\in\mathcal{B}} \partial B \setminus \partial\mathbb{D}$ with the constraint that it lies in M on $\bigcup_{B\in\mathcal{B}} \partial B \cap \partial\mathbb{D}$.

We aim at decreasing the energy of u as much as we can. We set

$$e_u = \sup\left\{ E(u) - E(H(u,\mathcal{B})); \mathcal{B} \in \mathcal{E}, \int_{\bigcup_{B\in\mathcal{B}} B} |\nabla u|^2 \leq \varepsilon_0 \right\},$$

so that if u is not already harmonic, we can pick some $\widetilde{\mathcal{B}}$ such that

$$E(u) - E(\tilde{u}) \geq \frac{e_u}{2} \tag{6.10}$$

where $\tilde{u} = H(u, \widetilde{\mathcal{B}})$ is the replaced map we choose. For this map, we can easily prove thanks to Theorem 59 in *STEP 2* and the definition of e_u that for any $\mathcal{B}$

$$\begin{aligned} \frac{1}{4}\int_{\mathbb{D}} |\nabla\tilde{u} - \nabla H(\tilde{u}, \frac{1}{2}\mathcal{B})|^2 &\leq E(\tilde{u}) - E(H(\tilde{u}, \frac{1}{2}\mathcal{B})) \\ &\leq E(u) - E(H(u,\mathcal{B})) + \frac{(E(u) - E(\tilde{u}))^{\frac{1}{2}}}{\kappa} \\ &\leq e_u + \frac{(E(u) - E(\tilde{u}))^{\frac{1}{2}}}{\kappa} \end{aligned}$$

and with the definition of $\tilde{u}$ (see (6.10)), we conclude that

$$\int_{\mathbb{D}} |\nabla\tilde{u} - \nabla H(\tilde{u}, \frac{1}{2}\mathcal{B})|^2 \leq C\,(E(u) - E(\tilde{u}))^{\frac{1}{2}} . \tag{6.11}$$

for any $\mathcal{B} \in \mathcal{E}$ with $\int_{\bigcup_{B\in\mathcal{B}} B} |\nabla u|^2 \leq \varepsilon_0$, up to decreasing ε_0 and for some constant C which only depends on M and N.

Therefore, if (u_n) is a minimizing sequence (we recall that we assumed that the set of parameters is trivial $\mathbb{B}^{k-2} = \{0\}$), we can define a new sequence as previously $\{\tilde{u}_n\}$ satisfying 6.11. By construction, $\{u_n\} \in \mathcal{A}$ and $E(\tilde{u}_n) \leq E(u_n)$ and by continuity of the harmonic replacement, by shrinking the radii of the balls and half balls used to define $\tilde{u}_n$, we see that u_n and $\tilde{u}_n$ lie in the same homotopy class. Therefore, $\{\tilde{u}_n\}$ is also a minimizing sequence so that thanks to 6.11,

$$\int_{\mathbb{D}} \left| \nabla\tilde{u}_n - \nabla H(\tilde{u}_n, \frac{1}{2}\mathcal{B}) \right|^2 \to 0 \text{ as } n \to +\infty$$

for any $\mathcal{B} \in \mathcal{E}$ such that $\int_{\bigcup_{B\in\mathcal{B}} B} |\nabla u_n|^2 \leq \varepsilon_0$.

Step 4 : A specific Palais-Smale assumption

Thanks to the previous steps, we can build a minimizing sequence of sweepouts $\sigma^n \in \omega$ such that for any sequence of parameters $t_n \in \mathbb{B}^{k-2}$ satisfying

$$\mathrm{Area}(u_n) \to W(N, M, \omega) \text{ as } n \to +\infty$$

with $u_n = \sigma^n_{t_n}$, we have the assumptions (6.12) and (6.13) of the following theorem implicitly proved in section 5 of **(9)**

Theorem 61 *Let $u_n \in \mathcal{A}$ be a sequence of maps with uniformly bounded energy such that*

$$\int_{\mathbb{D}} \left|\nabla u_n - \nabla H\left(u_n, \frac{1}{2}\mathcal{B}\right)\right|^2 \to 0 \text{ as } n \to +\infty \tag{6.12}$$

for any $\mathcal{B} \in \mathcal{E}$ such that $\int_{\bigcup_{B\in\mathcal{B}} B} |\nabla u_n|^2 \leq \varepsilon_0$ and

$$\mathrm{Area}(u_n) = E(u_n) + o(1) \text{ as } n \to +\infty\,. \tag{6.13}$$

Then, up to a subsequence, (u_n) verifies the $W^{1,2}$-bubble converges.

We can consider (6.12) as a kind of ε-regularity property for minimizing sequences of the energy. Indeed, outside a finite number of points where the energy could concentrate over ε_0, we deduce a strong convergence in $W^{1,2}$ to a harmonic map. (6.13) is a crucial assumption in order to get the no-neck energy. Usually, for harmonic equations, we use a Pohožaev identity to prove a no-neck energy lemma, see 3. Here, we do not have any equation but (6.13) is a quasi-conformal assumption, where being conformal is even stronger than a Pohožaev identity. It is why we can generalize Theorem 61 even in the case where u_n is defined on a set of degenerating conformal classes on a surface.

6.3 Energy convexity for biharmonic maps

For a short introduction to biharmonic maps we refer to section 3.6.

The main results of this section is an energy convexity for the intrinsic bi-harmonic energy. Developing the idea of Section 6.1 with Lin, we proved the following

Theorem 62 (Laurain-Lin 18, [L11]) *There exists a constant $\varepsilon_0 > 0$ such that if $u, v \in W^{2,2}(B_1, \mathbf{S}^n)$ with $u|_{\partial B_1} = v|_{\partial B_1}, \partial_\nu u|_{\partial B_1} = \partial_\nu v|_{\partial B_1}$,*

$$\int_{B_1} |\Delta u|^2\, dx = \int_{B_1} |\tau(u)|^2 + |\nabla u|^4\, dx \leq \varepsilon_0 \quad \text{and} \quad \int_{B_1} |\nabla v|^4\, dx \leq \varepsilon_0\,,$$

and u is a weakly intrinsic bi-harmonic map, then we have the energy convexity

$$\frac{1}{8}\int_{B_1} |\Delta v - \Delta u|^2\, dx \leq \frac{1}{2}\int_{B_1} |\tau(v) - \tau(u)|^2\, dx \leq \int_{B_1} |\tau(v)|^2\, dx - \int_{B_1} |\tau(u)|^2\, dx. \tag{6.14}$$

One of the benefits of the target manifold being $\mathbf{S}^n$ is that we automatically have $\int_{B_1} |\nabla u|^4\, dx \leq \int_{B_1} |\Delta u|^2\, dx$. In the general case, the control of the biharmonic-energy does not imply automatically the control of the gradient. But our result is still true if one assumes the smallness of the $W^{2,2}$ norm. Currently we try to get this estimate in order to prove the convexity in the general case.

Note also that the smallness condition $\int_{B_1} |\Delta u|^2\,dx = \int_{B_1} |\tau(u)|^2 + |\nabla u|^4\,dx \le \varepsilon_0$ is necessary : a harmonic map is an intrinsic bi-harmonic map as an absolute minimizer of the intrinsic bi-energy functional, but its Dirichlet energy $\int_{B_1} |\nabla u|^2\,dx$ could be large and the uniqueness is not possible, c.f. figure 6.1.

An immediate corollary of Theorem 62 is the uniqueness of weakly intrinsic biharmonic maps into spheres with small bi-energy on B_1.

Corollary 63 *There exists a constant $\varepsilon_0 > 0$ such that for all weakly intrinsic bi-harmonic maps (in particular, weakly harmonic maps) $u, v \in W^{2,2}(B_1, \mathbf{S}^n)$ with energies*

$$\int_{B_1} |\Delta u|^2\,dx \le \varepsilon_0\,, \quad \int_{B_1} |\Delta v|^2\,dx \le \varepsilon_0\,,$$

and $u|_{\partial B_1} = v|_{\partial B_1}, \partial_\nu u|_{\partial B_1} = \partial_\nu v|_{\partial B_1}$, we have $u \equiv v$ in B_1.

For extrinsic bi-harmonic maps, the convexity was announced in **[HHW14]**, but in fact they assume that $\int_{B_1} |\nabla^2 u|^2\,dx \le \varepsilon_0$, see remark 1.7 of **(10)**. Hence we gave a complete proof also for extrinsic bi-harmonic maps in **(10)**. Here is the statement of this result.

Theorem 64 (Theorem 2.2 of (10)) *There exists a constant $\varepsilon_0 > 0$ such that if $u, v \in W^{2,2}(B_1, \mathbf{S}^n)$ with $u|_{\partial B_1} = v|_{\partial B_1}, \partial_\nu u|_{\partial B_1} = \partial_\nu v|_{\partial B_1}$,*

$$\int_{B_1} |\Delta u|^2\,dx \le \varepsilon_0,,$$

and u is a weakly extrinsic bi-harmonic map, then we have the energy convexity

$$\frac{1}{2}\int_{B_1} |\Delta v - \Delta u|^2\,dx \le \int_{B_1} |\Delta v|^2\,dx - \int_{B_1} |\Delta u|^2\,dx. \tag{6.15}$$

Therefore, if additionally v is also a weakly extrinsic bi-harmonic map with $\int_{B_1} |\Delta v|^2\,dx \le \varepsilon_0,$, then $u \equiv v$ in B_1.

Thanks to the robustness of our method, which essentially relies on an improved ε-regularity result that follows from the technics developed by Lamm-Rivière **[LammR]** (see also **(4)**) and the idea developed in 6.1, we are able to prove a similar energy convexity along the intrinsic bi-harmonic heat flow into $\mathbf{S}^n$ treated as a perturbation of the stationary equation. One of the key observations of **(10)** is that such energy convexity along the flow yields the *coercivity* for the intrinsic bi-energy $I(u)$. Therefore, it proves the long time existence and uniform convergence of the intrinsic bi-harmonic map heat flow into $\mathbf{S}^n$ with small initial bi-energy. The corresponding results for the extrinsic bi-harmonic map heat flow with small initial bi-energy into general target manifolds were claimed in **[HHW14]** (which also follows directly from our method for spherical targets), but it seems that they need a smallness condition on the initial Hessian energy rather than the bi-energy in order to prove such results, see **[HHW14]**.

Theorem 65 (Laurain-Lin 18,(10)) *There exists a constant $\varepsilon_0 > 0$ such that if $u_0 \in C^\infty(\overline{B_1}, \mathbf{S}^n)$ with*

$$\int_{B_1} |\Delta u_0|^2\,dx \le \varepsilon_0\,, \tag{6.16}$$

then the initial-boundary value problem for the intrinsic bi-harmonic map heat flow (3.13) *has a smooth solution* $u \in C^\infty(\overline{B_1} \times [0,\infty), \mathbf{S}^n)$ *for all time. Moreover, there exists* $T_1 > 0$ *such that along the flow there holds an energy convexity*

$$\frac{1}{16}\int_{B_1} |\Delta u(\cdot,t_1) - \Delta u(\cdot,t_2)|^2 \leq \int_{B_1} |\tau(u)(\cdot,t_1)|^2 - \int_{B_1} |\tau(u)(\cdot,t_2)|^2 \tag{6.17}$$

for all $t_2 > t_1 \geq T_1$.

Immediate application of Theorem 65 is the following corollary.

Corollary 66 (Laurain-Lin 18,(10)) *There exists a constant* $\varepsilon_0 > 0$ *such that if* $u_0 \in C^\infty(\overline{B_1}, \mathbf{S}^n)$ *with* $\int_{B_1} |\Delta u_0|^2\, dx \leq \varepsilon_0$*, then the initial-boundary value problem for the intrinsic bi-harmonic map heat flow* (3.13) *has a smooth solution* $u \in C^\infty(\overline{B_1} \times [0,\infty), \mathbf{S}^n)$ *such that*

$$u(\cdot,t) \to u_\infty \text{ uniformly as } t \to +\infty \text{ strongly in } W^{2,2}(B_1, \mathbf{R}^{n+1})\,, \tag{6.18}$$

where u_∞ *is the unique smooth intrinsic bi-harmonic map with* $u_\infty|_{\partial B_1} = \chi$ *and* $\partial_\nu u_\infty|_{\partial B_1} = \xi$.

We expect that our result will have important consequences, as the energy convexity for weakly harmonic maps permitted Colding and Minicozzi to develop the *harmonic replacement* procedure, see the previous section, and prove the finite time extinction for certain 3-dimensional Ricci flow, see **[CM08]**. A replacement procedure involving bi-harmonic functions was also a key ingredient in the existence of Willmore minimizer by Simon, see **[Simon]**. Hence we can reasonably expect that our result will permit to perform a similar procedure in dimension 4.

Annexe A

Travaux de l’auteur

A.1 Travaux présentés dans ma thèse

(a) An obstruction to the existence of immersed curves of prescribed curvature, avec S.Kirsch, Potential Anal. 32 (2010), no. 1, 29-39.

(b) Concentration of CMC surfaces in a Riemannian manifold, 48 pages, Int. Math. Res. Not. IMRN 2012, no. 24, 5585–5649.

(c) Concentration of CMC surfaces with free boundaries, Annales de l’Institut Henri Poincare / Analyse non lineaire, Volume 29, Issue 1, January–February 2012, Pages 109-129.

A.2 Travaux présentés dans ce mémoire

(1) Stability of the Pohožaev obstruction in dimension 3, with O. Druet, J. Eur. Math. Soc. (JEMS) 12 (2010), no. 5, 1117–1149.

(2) Stability of elliptic PDEs with respect to perturbations of the domain, with O. Druet et E. Hebey, J. Differential Equations 255 (2013), no. 10, 3703–3718.

(3) Angular Energy Quantization for Linear Elliptic Systems with Antisymmetric Potentials and Applications, with T.Rivière, 38 pages, Anal. PDE 7 (2014), no. 1, 1–41.

(4) Energy Quantization for biharmonic maps, with T.Rivière, Advances in Calculus of Variations. Volume 6, Issue 2, Pages 191–216.

(5) A Pohožaev-type formula and Quantization of Horizontal Half-Harmonic Maps with F. Da Lio and T. Rivière, https ://arxiv.org/abs/1607.05504.

(6) Regularity and quantification for harmonic maps with free boundary, 16 pages, with R. Petrides, Adv. Calc. Var. 10 (2017), no. 1, 69–82.

(7) Energy quantization of Willmore surfaces at the boundary of the moduli space, with T. Rivi‘ere. Duke Math. J. 167 (2018), no. 11, 2073–2124.

(8) Optimal estimate for the gradient of Green functions on degenerating surfaces and applications, with T.Riviere, Comm. Anal. Geom. Vol. 26, No 4, 2018.

(9) Existence of min-max free boundary disks realizing the width of a manifold., with R. Petrides, Adv. Math. 352 (2019), 326–371.

(10) Energy convexity of intrinsic bi-harmonic maps and applications I : spherical target, with L. Lin, arXiv :1805.09428.

(11) Concentration of small Willmore spheres in Riemannian 3-manifolds, with A.Mondino, Anal. PDE 7 (2014), no. 8, 1901–1921.

(12) Uniqueness of Large Willmore surfaces in asymptotically flat spaces, with J. Metzger, In preparation.

(13) How to glue Willmore surfaces, with J. Lira, In preparation.

A.3 Autres Travaux

(1) Classification of uniformly distributed measures of dimension 1 in general co-dimension, with M. Petrache.

Table des figures

List of Symbols

Δ	Laplace operator	$\Delta = -\frac{\partial^2}{\partial^2 x_1} + \dots$
$\nabla^{\perp} f$	rotational in dimension 2	$-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}$
ν	the outward unit normal of a domain	
$f = O_k(r^{\alpha})$	O up to k derivatives	$\nabla^k f = O_k(r^{\alpha-k})$

Let $\vec{\Phi} : \Sigma \to \mathbb{R}^N$ a smooth immersion and $\vec{e}$ the map which to a point $p \in \Sigma$ assigns the oriented 2-plane given by the push forward by $\vec{\Phi}$ of the oriented tangent space $T_p\Sigma$. Using a positive orthonormal basis $(\vec{e}_1, \vec{e}_2)$ of $\vec{\Phi}_* T_p\Sigma$, we get

$$\vec{e} = \vec{e}_1 \wedge \vec{e}_2.$$

The Gauss map $\vec{n}$ assigns the oriented $N-2$-orthogonal plane to $\vec{e}$, that is to say

$$\vec{n} = \star\vec{e} = \vec{n}_1 \wedge \dots \wedge \vec{n}_{N-2},$$

where $\star$ is the Hodge operator on $\mathbb{R}^N$: if $\alpha \in \bigwedge^p \mathbb{R}^N$ then $\star\alpha \in \bigwedge^{N-p} \mathbb{R}^N$ such that for all $\beta \in \bigwedge^{N-p} \mathbb{R}^N$ we get

$$\beta \wedge \star\alpha = \langle \beta, \alpha \rangle \star 1,$$

where $\star 1$ is the canonical volume form of $\mathbb{R}^N$.

We will also need some other operator on $\bigwedge \mathbb{R}^N$. First, the contraction operator $\llcorner$: for every choice of $p-$, $q-$ and $p-q$ vectors, respectively α, β and γ the following holds

$$\langle \alpha \llcorner \beta, \gamma \rangle = \langle \alpha, \beta \wedge \gamma \rangle.$$

Thanks to this operator we can define the projection on the normal bundle as follow, for every $\vec{w} \in \mathbb{R}^N$ we set

$$\pi_{\vec{n}}(\vec{w}) = (-1)^{N-1} \vec{n} \llcorner (\vec{n} \llcorner \vec{w}).$$