STABILITY OF THE POHOŽAEV OBSTRUCTION IN DIMENSION 3

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Abstract. YYY

Definition 0.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^3$ and $h \in C^1(\mathbb{R}^3)$. We say that (Ω, h) satisfies the Pohožaev obstruction if Ω is star-shaped with respect to the origin and*

$$h(x) + \frac{1}{2}\langle x, \nabla h(x)\rangle \geq 0 \text{ for all } x \in \Omega. \tag{0.1}$$

Definition 0.2. *Let $\eta \in]0,1[$, Ω be a smooth bounded domain of $\mathbb{R}^3$ and $h \in C^1(\mathbb{R}^3)$. We say that (Ω, h) is $C^{1,\eta}$-stable if there exists $\delta(h, \Omega, \eta) > 0$ such that for all $h_\varepsilon \in C^{0,\eta}(R^3)$, $\Phi_\varepsilon \in C^{1,\eta} \cap \mathrm{Diff}(\mathbb{R}^3)$ and $\varphi_\varepsilon \in C^{0,\eta}(\partial\Omega)$, which satisfy*

$$\|h - h_\varepsilon\|_{C^{0,\eta}} + \|\Phi - Id\|_{C^{1,\eta}} + \|\varphi_\varepsilon\|_{C^{0,\eta}} \leq \delta$$

then the following problem has no solution,

$$\begin{cases} \Delta u + h_\varepsilon u = u^5 \text{ in } \Omega_\varepsilon \\ u > 0 \text{ in } \Omega_\varepsilon \\ u = \varphi_\varepsilon \circ \Phi_\varepsilon^{-1} \text{ on } \partial\Omega_\varepsilon \end{cases} \tag{0.2}$$

where $\Omega_\varepsilon = \Phi_\varepsilon(\Omega)$.

Theorem 1. *Let $\eta \in]0,1[$, Ω be a smooth bounded domain of $\mathbb{R}^3$ and $h \in C^1(\mathbb{R}^3)$ such that (Ω, h) satisfies the Pohožaev obstruction then (Ω, h) is always $C^{1,\eta}$-stable.*

The hypothesis of this theorem are optimal in the sense that we can find a Lipchitz perturbation of the ball which admits positve solutions, as pooved by the following theorem

Theorem 2. *There exists Φ_ε a smooth diffeomorphims of $\mathbb{R}^3$ closed to identity in any $C^{0,\eta}$ such that the following problem admits a solution*

$$\begin{cases} \Delta u = u^5 \text{ in } \Omega_\varepsilon \\ u > 0 \text{ in } \Omega_\varepsilon \\ u = \varphi_\varepsilon \text{ on } \partial\Omega_\varepsilon \end{cases} \tag{0.3}$$

where $\Omega_\varepsilon = \Phi_\varepsilon(B(0,1))$.

In fact, the technic work for any Ω which is axially symmetric and can probably be extended to any domain up to a technical effort to deal with the fact that the rescale domain is almost symmetric.

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1. Preliminaries

In fact a perturbation of the domain is equivalent to consider a fix domain with a perturbed metric. That is to say, if we get a solution u of

$$\begin{cases} \Delta u + h_\varepsilon u = u^5 \text{ in } \Omega_\varepsilon \\ u = \varphi_\varepsilon \circ \Phi_\varepsilon^{-1} \text{ on } \partial\Omega_\varepsilon \end{cases} \tag{1.1}$$

where $\Omega_\varepsilon = \Phi_\varepsilon(\Omega)$. Then $u_\varepsilon = u \circ \Phi_\varepsilon$ is a solution of

$$\begin{cases} \Delta_{g_\varepsilon} u_\varepsilon + \tilde{h}_\varepsilon u_\varepsilon = u^5 \text{ in } \Omega \\ u_\varepsilon = \varphi_\varepsilon \text{ on } \partial\Omega \end{cases} \tag{1.2}$$

where $\tilde{h}_\varepsilon = h_\varepsilon \circ \Phi_\varepsilon$ and $\Delta_{g_\varepsilon} = -\frac{1}{\sqrt{|g_\varepsilon|}}\partial_i \left(g_\varepsilon^{ij} \sqrt{|g_\varepsilon|} \partial_j \right)$ with $g_\varepsilon = \Phi_\varepsilon^*(\xi)$.

Then we are going to study the asymptotic behavior of sequences of solutions in this setting. We first prove the following theorem which is an equivalent of analysis done in section 2 of [3].

Theorem 3. *Let Ω a smooth bounded domain of $\mathbb{R}^3$, $h \in C^1(\mathbb{R}^3, \mathbb{R}_+)$, $\eta \in]0,1[$, $h_\varepsilon \in C^{0,\eta}(\mathbb{R}^3, \mathbb{R})$, $\Phi_\varepsilon \in C^{1,\eta} \cap \mathrm{Diff}(\mathbb{R}^3)$ and $\varphi_\varepsilon \in C^{0,\eta}(\partial\Omega, \mathbb{R})$ which satisfy*

$$\begin{gathered} \Phi_\varepsilon \to \mathrm{Id} \text{ in } C^{1,\eta}(\mathbb{R}^3), \\ h_\varepsilon \to h \text{ in } C^{0,\eta}(\overline{\Omega}), \\ \varphi_\varepsilon \to 0 \text{ in } C^{0,\eta}(\partial\Omega). \end{gathered} \tag{1.3}$$

Such that there exists $u_\varepsilon \in C^2(\Omega)$ a positive solution of

$$\begin{cases} \Delta_{g_\varepsilon} u_\varepsilon + \tilde{h}_\varepsilon u_\varepsilon = u^5 \text{ in } \Omega \\ u_\varepsilon = \varphi_\varepsilon \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where $g_\varepsilon = \Phi_\varepsilon^*(\xi)$. Then,there exists $u_0 \in C^{1,\eta}(\Omega)$ a nonegative solution of

$$\begin{cases} \Delta u + hu = u^5 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1.5}$$

there exist $N \in \mathbb{N}$, $x_1, \dots, x_N \in \Omega$ and $\lambda_1, \dots, \lambda_N \in \mathbb{R}_+^*$ such that, up to a subsequence,

$$\|u_\varepsilon\|_\infty u_\varepsilon(x) \to u_0 + \sum_{i=1}^N \lambda_i \mathcal{G}_h(x_i, x) \text{ in } C^1_{loc}(\Omega \setminus \{x_1, \dots, x_N\}) \text{as } \varepsilon \to 0$$

with $\mathcal{G}_h$ the Green function of the limit operator $\Delta + h$ with Dirichlet boundary condition on Ω.

2. Pointwise analysis around a concentration point

Let Ω be a smooth bounded domain of $\mathbb{R}^3$. We consider in this section some sequences $(h_\varepsilon), (\Phi_\varepsilon), (g_\varepsilon), (\varphi_\varepsilon)$ as in theorem 3 and a sequence (u_ε) of positive C^2-solutions of

$$\begin{cases} \Delta_{g_\varepsilon} u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^5 & \text{in } \Omega \\ u_\varepsilon = \varphi_\varepsilon & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

We also assume that we have a sequence (x_ε) of points in Ω and a sequence (ρ_ε) of positive real numbers with $0 < 3\rho_\varepsilon \leq d(x_\varepsilon, \partial\Omega)$ such that

$$\nabla u_\varepsilon(x_\varepsilon) = 0 \tag{2.2}$$

and

$$\rho_\varepsilon \left[\sup_{B(x_\varepsilon,\rho_\varepsilon)} u_\varepsilon(x) \right]^2 \to +\infty \text{ as } \varepsilon \to 0 \ . \tag{2.3}$$

We prove in this section that the following holds :

Proposition 2.1. *If there exists $C_0 > 0$ such that*

$$|x_\varepsilon - x|^{\frac{1}{2}} u_\varepsilon \leq C_0 \ \textit{in } B(x_\varepsilon, 3\rho_\varepsilon) \ , \tag{2.4}$$

then there exists $C_1 > 0$ such that

$$u_\varepsilon(x_\varepsilon) u_\varepsilon(x) \leq C_1 |x_\varepsilon - x|^{-1} \ \textit{in } B(x_\varepsilon, 2\rho_\varepsilon) \setminus \{x_\varepsilon\} \ \textit{and}$$
$$u_\varepsilon(x_\varepsilon) |\nabla u_\varepsilon(x)| \leq C_1 |x_\varepsilon - x|^{-2} \ \textit{in } B(x_\varepsilon, 2\rho_\varepsilon) \setminus \{x_\varepsilon\}.$$

Moreover, if $\rho_\varepsilon \to 0$, then

$$\rho_\varepsilon u_\varepsilon(x_\varepsilon) u_\varepsilon(x_\varepsilon + \rho_\varepsilon x) \to \frac{1}{|x|} + b \ \textit{in } C^1_{loc}(B(0,2) \setminus \{0\}) \ \textit{as } \varepsilon \to 0$$

where b is some harmonic function in $B(0,2)$ with $b(0) = 0$ and $\nabla b(0) = 0$.

The rest of this section is dedicated to the proof of this proposition. We follow step by step [3].

We divide the proof of the proposition into several claims. The first one gives the asymptotic behaviour of u_ε around x_ε at an appropriate small scale.

Claim 2.1. *After passing to a subsequence, we have that*

$$\mu_\varepsilon^{\frac{1}{2}} u_\varepsilon(x_\varepsilon + \mu_\varepsilon x) \to \frac{1}{\left(1 + \frac{|x|^2}{3}\right)^{\frac{1}{2}}} \ \textit{in } C^1_{loc}(\mathbb{R}^3), \ \textit{as } \varepsilon \to 0 \tag{2.5}$$

where $\mu_\varepsilon = u_\varepsilon(x_\varepsilon)^{-2}$.

Proof of Claim 2.1. Let $\tilde{x}_\varepsilon \in \overline{B(x_\varepsilon, \rho_\varepsilon)}$ and $\tilde{\mu}_\varepsilon > 0$ be such that

$$u_\varepsilon(\tilde{x}_\varepsilon) = \sup_{B(x_\varepsilon,\rho_\varepsilon)} u_\varepsilon = \tilde{\mu}_\varepsilon^{-\frac{1}{2}} \ . \tag{2.6}$$

Thanks to (2.3), we have that

$$\tilde{\mu}_\varepsilon \to 0 \text{ and } \frac{\rho_\varepsilon}{\tilde{\mu}_\varepsilon} \to +\infty \text{ as } \varepsilon \to 0 \ . \tag{2.7}$$

Thanks to (2.4), we also have that

$$|x_\varepsilon - \tilde{x}_\varepsilon| = O(\tilde{\mu}_\varepsilon). \tag{2.8}$$

We set for $x \in \Omega_\varepsilon = \left\{ x \in \mathbb{R}^3 \text{ s.t. } \tilde{x}_\varepsilon + \tilde{\mu}_\varepsilon x \in \Omega \right\}$,

$$\tilde{u}_\varepsilon(x) = \tilde{\mu}_\varepsilon^{\frac{1}{2}} u_\varepsilon(\tilde{x}_\varepsilon + \tilde{\mu}_\varepsilon x)$$

which verifies

$$\begin{aligned} &\Delta_{\tilde{g}_\varepsilon} \tilde{u}_\varepsilon + \tilde{\mu}_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon = \tilde{u}_\varepsilon^5 \text{ in } \Omega_\varepsilon \ , \\ &\tilde{u}_\varepsilon(0) = \sup_{B(\frac{x_\varepsilon - \tilde{x}_\varepsilon}{\tilde{\mu}_\varepsilon}, \frac{\rho_\varepsilon}{\tilde{\mu}_\varepsilon})} \tilde{u}_\varepsilon = 1 \ , \end{aligned} \tag{2.9}$$

where $\tilde{h}_\varepsilon = h(\tilde{x}_\varepsilon + \tilde{\mu}_\varepsilon x)$ and $\tilde{g}_\varepsilon = g_\varepsilon(\tilde{x}_\varepsilon + \tilde{\mu}_\varepsilon x)$. Thanks to (2.7) and (2.8), we get that

$$B\left(\frac{x_\varepsilon - \tilde{x}_\varepsilon}{\tilde{\mu}_\varepsilon}, \frac{\rho_\varepsilon}{\tilde{\mu}_\varepsilon}\right) \to \mathbb{R}^3 \text{ as } \varepsilon \to 0 \,. \tag{2.10}$$

Now, thanks to (2.9), (2.10), and by standard elliptic theory (Theorem 3.13 of [4]), we get that, after passing to a subsequence, $\tilde{u}_\varepsilon \to U$ in $C^1_{loc}(\mathbb{R}^3)$ as $\varepsilon \to 0$ where U satisfies

$$\Delta U = U^5 \text{ in } \mathbb{R}^3 \text{ and } 0 \le U \le 1 = U(0) \,.$$

Thanks to the work of Caffarelli, Gidas and Spruck [1], we know that

$$U(x) = \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}} \,.$$

Moreover, thanks to (2.8), we know that, after passing to a new subsequence, $\frac{x_\varepsilon - \tilde{x}_\varepsilon}{\tilde{\mu}_\varepsilon} \to x_0$ as $\varepsilon \to 0$ for some $x_0 \in \mathbb{R}^3$. Hence, since x_ε is a critical point of u_ε, x_0 must be a critical point of U, namely $x_0 = 0$. We deduce that $\frac{\mu_\varepsilon}{\tilde{\mu}_\varepsilon} \to 1$ where μ_ε is as in the statement of the claim. The claim 2.1 follows. ■

For $0 \le r \le 3\rho_\varepsilon$, we set

$$\psi_\varepsilon(r) = \frac{r^{\frac{1}{2}}}{\omega_2 r^2} \int_{\partial B(x_\varepsilon, r)} u_\varepsilon d\sigma$$

where $d\sigma$ denotes the Lebesgue measure on the sphere $\partial B(x_\varepsilon, r)$ and $\omega_2 = 4\pi$ is the volume of the unit 2-sphere. We easily check, thanks to Claim 2.1, that

$$\psi_\varepsilon(\mu_\varepsilon r) = \left(\frac{r}{1 + \frac{r^2}{3}}\right)^{\frac{1}{2}} + o(1),\ \psi'_\varepsilon(\mu_\varepsilon r) = \frac{1}{2}\left(\frac{r}{1 + \frac{r^2}{3}}\right)^{\frac{3}{2}} \left(\frac{1}{r^2} - \frac{1}{3}\right) + o(1) \,. \tag{2.11}$$

We define r_ε by

$$r_\varepsilon = \max\left\{ r \in [2\sqrt{3}\mu_\varepsilon, \rho_\varepsilon] \text{ s.t. } \psi'_\varepsilon(s) \le 0 \text{ for } s \in [2\sqrt{3}\mu_\varepsilon, r] \right\} \,.$$

Thanks to (2.11), the set on which the maximum is taken is not empty for ε small enough, and moreover

$$\frac{r_\varepsilon}{\mu_\varepsilon} \to +\infty \text{ as } \varepsilon \to 0 \,. \tag{2.12}$$

We prove now the following :

Claim 2.2. *There exists $C > 0$, independent of ε, such that*

$$u_\varepsilon(x) \le C\mu_\varepsilon^{\frac{1}{2}} |x_\varepsilon - x|^{-1} \text{ in } B(x_\varepsilon, 2r_\varepsilon) \setminus \{x_\varepsilon\},$$
$$|\nabla u_\varepsilon(x)| \le C\mu_\varepsilon^{\frac{1}{2}} |x_\varepsilon - x|^{-2} \text{ in } B(x_\varepsilon, 2r_\varepsilon) \setminus \{x_\varepsilon\} \text{ and}$$
$$|\nabla^2 u_\varepsilon(x)| \le C\mu_\varepsilon^{\frac{1}{2}} |x_\varepsilon - x|^{-3} \text{ in } B(x_\varepsilon, 2r_\varepsilon) \setminus \{x_\varepsilon\} \,.$$

Proof of Claim 2.2. Here we follow the proof of Lemma 1.5 and 1.6 of [2] since contrary to [3] we get a pointwise convergence of h_ε to h. We first prove that for any given $0 < \nu < \frac{1}{2}$, there exists $C_\nu > 0$ such that

$$u_\varepsilon(x) \le C_\nu \left(\mu_\varepsilon^{\frac{1}{2}(1-2\nu)} |x - x_\varepsilon|^{-(1-\nu)} + \alpha_\varepsilon \left(\frac{r_\varepsilon}{|x - x_\varepsilon|}\right)^\nu \right) \tag{2.13}$$

for all $x \in B(x_\varepsilon, 2r_\varepsilon)$ and ε small enough, where

$$\alpha_\varepsilon = \left(\sup_{\partial B(x_\varepsilon, r_\varepsilon)} u_\varepsilon \right) . \tag{2.14}$$

First of all, we can use (2.4) and apply the Harnack inequality, see for instance theorem 4.17 of [4], to get the existence of some $C > 0$ such that

$$\frac{1}{C} \max_{\partial B(x_\varepsilon, r)} (u_\varepsilon + r|\nabla u_\varepsilon|) \leq \frac{1}{\omega_2 r^2} \int_{\partial B(x_\varepsilon, r)} u_\varepsilon d\sigma \leq C \min_{\partial B(x_\varepsilon, r)} u_\varepsilon \tag{2.15}$$

for all $0 < r < \frac{5}{2}\rho_\varepsilon$ and all $\varepsilon > 0$. The details of the proof of such an assertion may be found in [2], lemma 1.3. Hence, thanks to (2.11) and (2.12), we have that

$$|x - x_\varepsilon|^{\frac{1}{2}} u_\varepsilon(x) \leq C\psi_\varepsilon(r) \leq C\psi_\varepsilon(R\mu_\varepsilon) = C \left(\frac{R}{1 + \frac{R^2}{3}} \right)^{\frac{1}{2}} + o(1)$$

for all $R \geq 2\sqrt{3}$, all $r \in [R\mu_\varepsilon, r_\varepsilon]$, all ε small enough and all $x \in \partial B(x_\varepsilon, r)$. Thus we get that

$$\sup_{B(x_\varepsilon, r_\varepsilon) \setminus B(x_\varepsilon, R\mu_\varepsilon)} |x - x_\varepsilon|^{\frac{1}{2}} u_\varepsilon(x) = e(R) + o(1) \tag{2.16}$$

where $e(R) \to 0$ as $R \to +\infty$. Let $\mathcal{G}_\varepsilon$ be the Green function of the operator Δ_{g_ε} in Ω with Dirichlet boundary condition. Then, see for instance [**?**] , we get the existence of some $C > 0$ such that

$$\left| |x - y| \mathcal{G}_\varepsilon(x, y) - \frac{1}{\omega_2} \right| \leq C|x - y|, \tag{2.17}$$

and that

$$\left| |x - y|^2 |\nabla \mathcal{G}_\varepsilon(x, y)| - \frac{1}{\omega_2} \right| \leq C|x - y|, \tag{2.18}$$

for all $x \neq y \in \Omega$. We fix $0 < \nu < \frac{1}{2}$ and we set

$$\Phi_{\varepsilon,\nu} = \mu_\varepsilon^{\frac{1}{2}(1-2\nu)} \mathcal{G}_\varepsilon(x_\varepsilon, x)^{1-\nu} + \alpha_\varepsilon \big(r_\varepsilon \mathcal{G}_\varepsilon(x_\varepsilon, x) \big)^\nu .$$

Thanks to (2.17), (2.13) reduces to prove that

$$\sup_{B(x_\varepsilon, 2r_\varepsilon)} \frac{u_\varepsilon}{\Phi_{\varepsilon,\nu}} = O(1) .$$

We let $y_\varepsilon \in \overline{B(x_\varepsilon, 2r_\varepsilon) \setminus \{x_\varepsilon\}}$ be such that

$$\sup_{B(x_\varepsilon, 2r_\varepsilon)} \frac{u_\varepsilon}{\Phi_{\varepsilon,\nu}} = \frac{u_\varepsilon(y_\varepsilon)}{\Phi_{\varepsilon,\nu}(y_\varepsilon)} .$$

We are going to consider the several possible beahviour of the sequence (y_ε).

First of all, assume that

$$\frac{|x_\varepsilon - y_\varepsilon|}{\mu_\varepsilon} \to R \text{ as } \varepsilon \to 0 .$$

Thanks to Claim 2.1, we have in this case that

$$\mu_\varepsilon^{\frac{1}{2}} u_\varepsilon(y_\varepsilon) \to (1 + R^2)^{-\frac{1}{2}} \text{ as } \varepsilon \to 0.$$

On the other hand, thanks to (2.16), we can write that

$$\begin{aligned}\mu_\varepsilon^{\frac{1}{2}}\Phi_{\varepsilon,\nu}(y_\varepsilon) &= \left(\frac{\mu_\varepsilon}{\omega_2|x_\varepsilon-y_\varepsilon|}\right)^{1-\nu}+O\left(\alpha_\varepsilon\mu_\varepsilon^{\frac{1}{2}}\left(\frac{r_\varepsilon}{|x_\varepsilon-y_\varepsilon|}\right)^{\nu}\right)+o(1)\\ &= (R\omega_2)^{\nu-1}+O\left((r_\varepsilon^{\frac{1}{2}}\alpha_\varepsilon)\mu_\varepsilon^{\frac{1}{2}(1-2\nu)}r_\varepsilon^{\frac{1}{2}(2\nu-1)}\right)+o(1)\\ &= (R\omega_2)^{\nu-1}+o(1),\end{aligned}$$

if $R>0$, and $\mu_\varepsilon^{\frac{1}{2}}\Phi_{\varepsilon,\nu}(y_\varepsilon)\to+\infty$ as $\varepsilon\to 0$ if $R=0$. In any case, $\left(\frac{u_\varepsilon(y_\varepsilon)}{\Phi_{\varepsilon,\nu(y_\varepsilon)}}\right)$ is bounded.

Assume now that there exists $\delta>0$ such that $y_\varepsilon\in B(y_\varepsilon,r_\varepsilon)\setminus B(y_\varepsilon,\delta r_\varepsilon)$. Thanks to Harnack's inequality (2.15), we get that $u_\varepsilon(y_\varepsilon)=O(\alpha_\varepsilon)$ which, thanks to (2.17), easily gives that $\frac{u_\varepsilon(y_\varepsilon)}{\Phi_{\varepsilon,\nu(y_\varepsilon)}}=O(1)$.

Hence, we are left with the following situation :

$$\frac{|y_\varepsilon-x_\varepsilon|}{r_\varepsilon}\to 0 \text{ and } \frac{|x_\varepsilon-y_\varepsilon|}{\mu_\varepsilon}\to+\infty \text{ as } \varepsilon\to 0\ . \tag{2.19}$$

Thanks to the definition of y_ε, we can then write that

$$\frac{\Delta_{g_\varepsilon}u_\varepsilon(y_\varepsilon)}{u_\varepsilon(y_\varepsilon)}\geq\frac{\Delta_{g_\varepsilon}\Phi_{\varepsilon,\nu}(y_\varepsilon)}{\Phi_{\varepsilon,\nu}(y_\varepsilon)}$$

which gives, thanks to the definition of $\Phi_{\varepsilon,\nu}$ and multiplying by $|x_\varepsilon-y_\varepsilon|^2$, that

$$\begin{aligned}|x_\varepsilon-y_\varepsilon|^2u_\varepsilon(y_\varepsilon)^4 \geq\ & |x_\varepsilon-y_\varepsilon|h_\varepsilon(y_\varepsilon)+\nu(1-\nu)\frac{|x_\varepsilon-y_\varepsilon|^2}{\Phi_{\varepsilon,\nu}(y_\varepsilon)}\left(\alpha_\varepsilon r_\varepsilon^\nu\frac{|\nabla_{g_\varepsilon}\mathcal{G}_\varepsilon(x_\varepsilon,y_\varepsilon)|^2_{g_\varepsilon}}{\mathcal{G}_\varepsilon(x_\varepsilon,y_\varepsilon)^2}\mathcal{G}_\varepsilon(x_\varepsilon,y_\varepsilon)^\nu\right.\\ &\left.+\mu_\varepsilon^{\frac{1}{2}(1-2\nu)}\frac{|\nabla_{g_\varepsilon}\mathcal{G}_\varepsilon(x_\varepsilon,y_\varepsilon)|^2_{g_\varepsilon}}{\mathcal{G}_\varepsilon(x_\varepsilon,y_\varepsilon)^2}\mathcal{G}_\varepsilon(x_\varepsilon,y_\varepsilon)^{1-\nu}\right)\ .\end{aligned}$$

Thanks to (2.16), the left-hand side goes to 0 as $\varepsilon\to 0$. Then, thanks to (2.17), (2.18) and (2.19), we get that

$$o(1)\geq\nu(1-\nu)+o(1)$$

which is a contradiction, and shows that this last case can not occur. This ends the proof of (2.13).

We now claim that there exists $C>0$, independent of ε, such that

$$u_\varepsilon(x)\leq C\left(\mu_\varepsilon^{\frac{1}{2}}|x-x_\varepsilon|^{-1}+\alpha_\varepsilon\right) \text{ in } B(x_\varepsilon,r_\varepsilon)\ . \tag{2.20}$$

Thanks to Claim 2.1 and (2.15), this holds for all sequences $y_\varepsilon\in B(x_\varepsilon,r_\varepsilon)\setminus\{x_\varepsilon\}$ such that $|y_\varepsilon-x_\varepsilon|=O(\mu_\varepsilon)$ or $\frac{|y_\varepsilon-x_\varepsilon|}{r_\varepsilon}\not\to 0$. Thus we may assume from now that

$$\frac{|y_\varepsilon-x_\varepsilon|}{\mu_\varepsilon}\to+\infty \text{ and } \frac{|y_\varepsilon-x_\varepsilon|}{r_\varepsilon}\to 0 \text{ as } \varepsilon\to 0\ .$$

Thanks to the Green representation formula, we write with (2.17) and (2.18) that

$$\begin{aligned}u_\varepsilon(y_\varepsilon) &= \int_{B(x_\varepsilon,r_\varepsilon)}\tilde{\mathcal{G}}_\varepsilon\Delta_{g_\varepsilon}u_\varepsilon\,dx\\ &\quad+O\left(r_\varepsilon^{-2}\int_{\partial B(x_\varepsilon,r_\varepsilon)}u_\varepsilon\,d\sigma\right)\ .\end{aligned}$$

Here $\tilde{\mathcal{G}}_\varepsilon$ is the Green function of the operator $\Delta_{g_\varepsilon} + h_\varepsilon$ in $B(x_\varepsilon, r_\varepsilon)$. Indeed this operator is coercive since $h \geq 0$. It also satisfies classical estimate as (2.17) and (2.18). This gives with (2.14), (2.15) and (2.17) that

$$u_\varepsilon(y_\varepsilon) = O\left(\int_{B(x_\varepsilon,r_\varepsilon)} |x-y_\varepsilon|^{-1} \left|\Delta_{g_\varepsilon} u_\varepsilon + h_\varepsilon u_\varepsilon\right| dx\right) + O\left(\alpha_\varepsilon\right) \ . \tag{2.21}$$

Using (2.13) with $\nu = \frac{1}{5}$, we can write that

$$\begin{aligned}
&\int_{B(x_\varepsilon,r_\varepsilon)} |x-y_\varepsilon|^{-1} \left|\Delta_{g_\varepsilon} u_\varepsilon + h_\varepsilon u_\varepsilon\right| dx \\
&= \int_{B(x_\varepsilon,\mu_\varepsilon)} \frac{u_\varepsilon^5}{|x-y_\varepsilon|} dx + \int_{B(x_\varepsilon,r_\varepsilon)\setminus B(x_\varepsilon,\mu_\varepsilon)} |x-y_\varepsilon|^{-1} u_\varepsilon^5 dx \\
&= O\left(\mu_\varepsilon^{\frac{1}{2}} |y_\varepsilon - x_\varepsilon|^{-1}\right) + \alpha_\varepsilon^5 r_\varepsilon \int_{B(x_\varepsilon,r_\varepsilon)\setminus B(x_\varepsilon,\mu_\varepsilon)} |x-y_\varepsilon|^{-1} |x-x_\varepsilon|^{-1} dx \\
&\quad + \mu_\varepsilon^{\frac{3}{2}} \int_{B(x_\varepsilon,r_\varepsilon)\setminus B(x_\varepsilon,\mu_\varepsilon)} |x-y_\varepsilon|^{-1} |x-x_\varepsilon|^{-4} dx \\
&= O\left(\mu_\varepsilon^{\frac{1}{2}} |y_\varepsilon - x_\varepsilon|^{-1}\right) + O(\alpha_\varepsilon^5 r_\varepsilon^2) \ .
\end{aligned}$$

Thanks to (2.12) and to (2.16), this leads to

$$\int_{B(x_\varepsilon,r_\varepsilon)} |x-y_\varepsilon|^{-1} \left|\Delta_{g_\varepsilon} u_\varepsilon + h_\varepsilon u_\varepsilon\right| dx \leq O(\mu_\varepsilon^{\frac{1}{2}} |y_\varepsilon - x_\varepsilon|^{-1}) + o(\alpha_\varepsilon),$$

which, thanks to (2.21), proves (2.20).

In order to end the proof of the first part of the claim, we just have to prove that

$$\alpha_\varepsilon = \sup_{\partial B(x_\varepsilon,r_\varepsilon)} u_\varepsilon = O\left(\mu_\varepsilon^{\frac{1}{2}} r_\varepsilon^{-1}\right) \ . \tag{2.22}$$

For that purpose, we use the definition of r_ε to write that

$$(\beta r_\varepsilon)^{\frac{1}{2}} \psi_\varepsilon(\beta r_\varepsilon) \geq (r_\varepsilon)^{\frac{1}{2}} \psi_\varepsilon(r_\varepsilon)$$

for all $0 < \beta < 1$. Using (2.15), this leads to

$$r_\varepsilon^{\frac{1}{2}} \left(\sup_{\partial B(x_\varepsilon,r_\varepsilon)} u_\varepsilon\right) \leq C(\beta r_\varepsilon)^{\frac{1}{2}} \left(\sup_{\partial B(x_\varepsilon,\beta r_\varepsilon)} u_\varepsilon\right) \ .$$

Thanks to (2.20), we obtain that

$$\left(\sup_{\partial B(x_\varepsilon,r_\varepsilon)} u_\varepsilon\right) \leq C\beta^{\frac{1}{2}} \left(\mu_\varepsilon^{\frac{1}{2}} (\beta r_\varepsilon)^{-1} + \sup_{\partial B(x_\varepsilon,r_\varepsilon)} u_\varepsilon\right) \ .$$

Choosing β small enough clearly gives (2.22) and thus the pointwise estimate on u_ε of the claim. The estimates on ∇u_ε and $\nabla^2 u_\varepsilon$ follow from standard elliptic theory.■

We now prove the following :

Claim 2.3. *If $r_\varepsilon \to 0$ as $\varepsilon \to 0$, up to passing to a subsequence,*

$$r_\varepsilon u_\varepsilon(x_\varepsilon) u_\varepsilon(x_\varepsilon + r_\varepsilon x) \to \frac{1}{|x|} + b \text{ in } C^1_{loc}\left(B\left(0,2\right) \setminus \{0\}\right) \text{ as } \varepsilon \to 0$$

where b is some harmonic function in $B(0,2)$. Moreover, if $r_\varepsilon < \rho_\varepsilon$, then $b(0) = 1$.

Proof of Claim 2.3. We set, for $x \in B(0,2)$,

$$\tilde{u}_\varepsilon(x) = \mu_\varepsilon^{-\frac{1}{2}} r_\varepsilon u_\varepsilon(x_\varepsilon + r_\varepsilon x)$$

which verifies

$$\Delta_{\tilde{g}_\varepsilon} \tilde{u}_\varepsilon + r_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon = \left(\frac{\mu_\varepsilon}{r_\varepsilon}\right)^2 \tilde{u}_\varepsilon^5 \text{ in } B(0,2) \tag{2.23}$$

where $\tilde{h}_\varepsilon = h(x_\varepsilon + r_\varepsilon x)$ and $\tilde{g}_\varepsilon = g(x_\varepsilon + r_\varepsilon x)$. Thanks to Claim 2.2, there exists $C > 0$ such that

$$\tilde{u}_\varepsilon(x) \leq \frac{C}{|x|} \text{ in } B(0,2) \setminus \{0\}. \tag{2.24}$$

Then, thanks to standard elliptic theory, we get that, after passing to a subsequence, $\tilde{u}_\varepsilon \to U$ in $C^1_{loc}(B(0,2) \setminus \{0\})$ as $\varepsilon \to 0$ where U is a non-negative solution of

$$\Delta U = 0 \text{ in } B(0,2) \setminus \{0\} .$$

Then, thanks to the Bôcher theorem on singularities of harmonic functions, we get that

$$U(x) = \frac{\lambda}{|x|} + b(x)$$

where b is some harmonic function in $B(0,2)$ and $\lambda \geq 0$. Now, integrating (2.23) on $B(0,1)$ with respect to $\tilde{g}_\varepsilon$, we get that

$$\int_{\partial B(0,1)} \partial_{\tilde{\nu}_\varepsilon} \tilde{u}_\varepsilon d\sigma_{\tilde{g}_\varepsilon} = \int_{B(0,1)} \left(r_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon - \left(\frac{\mu_\varepsilon}{r_\varepsilon}\right)^2 \tilde{u}_\varepsilon^5 \right) dv_{\tilde{g}_\varepsilon},$$

where $\tilde{\nu}_\varepsilon$ is the exterior normal to $\partial B(0,1)$ with respect to $\tilde{g}_\varepsilon$.Thanks to Claim 2.2,

$$\int_{B(0,1)} r_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon \, dv_{\tilde{g}_\varepsilon} \to 0 \text{ as } \varepsilon \to 0$$

and, thanks to Claim 2.1,

$$\int_{B(0,1)} \left(\frac{\mu_\varepsilon}{r_\varepsilon}\right)^2 \tilde{u}_\varepsilon^5 dv_{\tilde{g}_\varepsilon} \to \int_{\mathbb{R}^3} \left(1 + \frac{|x|^2}{3}\right)^{-\frac{5}{2}} dx = \omega_2 \text{ as } \varepsilon \to 0 .$$

On the other hand, we have that

$$\int_{\partial B(0,1)} \partial_{\tilde{\nu}_\varepsilon} \tilde{u}_\varepsilon \, d\sigma_{\tilde{g}_\varepsilon} \to -\omega_2 \lambda \text{ as } \varepsilon \to 0 .$$

We deduce that $\lambda = 1$, which proves the first part of the Claim.

Now, if $r_\varepsilon < \rho_\varepsilon$, we have thanks to the definition of r_ε that

$$\psi'_\varepsilon(r_\varepsilon) = 0 .$$

Setting $\tilde{\psi}_\varepsilon(r) = \left(\frac{r_\varepsilon}{\mu_\varepsilon}\right)^{\frac{1}{2}} \psi_\varepsilon(r_\varepsilon r)$ for $0 < r < 2$, we see that

$$\tilde{\psi}_\varepsilon(r) \to \frac{r^{\frac{1}{2}}}{\omega_2 r^2} \int_{\partial B(0,r)} U \, d\sigma = r^{-\frac{1}{2}} + r^{\frac{1}{2}} b(0) .$$

We deduce that $b(0) = 1$, which ends the proof of the Claim. ■

We prove at last the following :

Claim 2.4. *Using the notations of Claim 2.3, we have that $b(0) = 0$ and $\nabla b(0) = 0$.*

Proof of Claim 2.4. We use the notation of the proof of Claim 2.3. Let us apply the Pohožaev identity (A.1) of appendix A.2 to $\tilde{u}_\varepsilon$ in $B(0,1)$, In fact we will apply it to $u^\varepsilon \circ \Phi_\varepsilon^{-1}$ on $\Phi_\varepsilon(B(\Phi_\varepsilon(0),1))$ and then we pull it back on $B(0,1)$, which gives

$$\frac{1}{2}\int_{B(0,1)} r_\varepsilon^2 \left(\tilde{h}_\varepsilon \tilde{u}_\varepsilon^2 + \tilde{h}_\varepsilon \langle \Phi_\varepsilon(x), \nabla(\tilde{u}_\varepsilon \circ \Phi_\varepsilon^{-1})^2 \circ \Phi_\varepsilon\rangle\right) dv_{\tilde{g}_\varepsilon} = \tilde{B}^\varepsilon \tag{2.25}$$

where

$$\tilde{B}^\varepsilon = \int_{\partial B(0,1)} \langle \Phi_\varepsilon(x), \nabla(\tilde{u}_\varepsilon \circ \Phi_\varepsilon^{-1}) \circ \Phi_\varepsilon\rangle \left(\partial_{\nu_\varepsilon} \tilde{u}_\varepsilon\right) + \frac{1}{2}\tilde{u}_\varepsilon \partial_{\nu_\varepsilon}\tilde{u}_\varepsilon - \frac{\langle \Phi_\varepsilon(x), \overline{\nu}_\varepsilon \circ \Phi_\varepsilon\rangle}{2} |\nabla(\tilde{u}_\varepsilon \circ \Phi_\varepsilon^{-1}) \circ \Phi_\varepsilon|^2 d\sigma_\varepsilon$$
$$+ \int_{\partial B(0,1)} \langle \Phi_\varepsilon(x), \overline{\nu}_\varepsilon \circ \Phi_\varepsilon\rangle \left(\frac{\mu_\varepsilon}{r_\varepsilon}\right)^2 \frac{\tilde{u}_\varepsilon^6}{6} d\sigma_\varepsilon,$$

where $\overline{\nu}_\varepsilon$ is the exterior normal of $\Phi_\varepsilon(\partial B(0,1))$ and ν_ε and $d\sigma_\varepsilon$ are respectively the exterior normal and the volume element induced by $\tilde{g}_\varepsilon$ on $\partial B(0,1)$.

Thanks to Claim 2.2 and the fact that Φ_ε goes to Id in $C^{1,\eta}$, we can pass to the limit to obtain that

$$\int_{\partial B(0,1)} (\partial_\nu U)^2 + \frac{1}{2} U \partial_\nu U - \frac{|\nabla U|^2}{2} d\sigma = 0 \ .$$

Since b is harmonic, it is easily checked that the left-hand side is just $-\frac{\omega_2 b(0)}{2}$. This proves that $b(0) = 0$.

In order to prove the second part of the claim, we apply the Pohožaev identity (A.4) of appendix A.2 to $\tilde{u}_\varepsilon$ in $B(0,1)$. We obtain that

$$\begin{aligned} &\int_{\partial B(0,1)} \left(\frac{|\nabla(\tilde{u}_\varepsilon \circ \Phi_\varepsilon^{-1}) \circ \Phi_\varepsilon|^2}{2} \overline{\nu}_\varepsilon \circ \Phi_\varepsilon^{-1} - \partial_{\nu_\varepsilon} \tilde{u}_\varepsilon \nabla\left(\tilde{u}_\varepsilon \circ \Phi_\varepsilon^{-1}\right) \circ \Phi_\varepsilon\right) d\sigma_\varepsilon \\ &= -\int_{B(0,1)} r_\varepsilon^2 \tilde{h}_\varepsilon \frac{\nabla\left(\tilde{u}_\varepsilon \circ \Phi_\varepsilon^{-1}\right)^2 \circ \Phi_\varepsilon}{2} dv_{\tilde{g}_\varepsilon} - \int_{\partial B(0,1)} \left(\frac{\mu_\varepsilon}{r_\varepsilon}\right)^2 \frac{\tilde{u}_\varepsilon^6}{6} \overline{\nu}_\varepsilon \circ \Phi^\varepsilon \, d\sigma_\varepsilon \ , \end{aligned} \tag{2.26}$$

where $\overline{\nu}_\varepsilon$ is the exterior normal of $\Phi_\varepsilon(\partial B(0,1))$. Thanjs to the fact that Φ_ε goes to Id in $C^{1,\eta}$, we can pass to the limit to obtain that

$$\int_{\partial B(0,1)} \left(\frac{|\nabla(\tilde{u}_\varepsilon \circ \Phi_\varepsilon^{-1}) \circ \Phi_\varepsilon|^2}{2} \overline{\nu}_\varepsilon \circ \Phi_\varepsilon^{-1} - \partial_{\nu_\varepsilon} \tilde{u}_\varepsilon \nabla\left(\tilde{u}_\varepsilon \circ \Phi_\varepsilon^{-1}\right) \circ \Phi_\varepsilon\right) d\sigma_\varepsilon \to \int_{\partial B(0,1)} \left(\frac{|\nabla U|^2}{2}\nu - \partial_\nu U \nabla U\right) d\sigma.$$

Moreover, thanks to the fact that b is harmonic, we easily get that

$$\int_{\partial B(0,1)} \left(\frac{|\nabla U|^2}{2}\nu - \nabla U \partial_\nu U\right) d\sigma = \omega_2 \nabla b(0) \ .$$

It remains to deals with the right-hand side of (2.26). It is clear that

$$\int_{\partial B(0,1)} \left(\frac{\mu_\varepsilon}{r_\varepsilon}\right)^2 \frac{\tilde{u}_\varepsilon^6}{6} \overline{\nu}_\varepsilon \circ \Phi_\varepsilon d\sigma_\varepsilon \to 0 \text{ as } \varepsilon \to 0 \ .$$

Then we rewrite the first term of the right-hand side of (2.26) as

$$\int_{B(0,1)} r_\varepsilon^2 \tilde{h}_\varepsilon \frac{\nabla \left(\tilde{u}_\varepsilon \circ \Phi_\varepsilon^{-1}\right)^2 \circ \Phi_\varepsilon}{2} \, dv_{\tilde{g}_\varepsilon} = \int_{B(0,1)} r_\varepsilon^2 (\tilde{h}_\varepsilon - \tilde{h}_\varepsilon(0)) \frac{\nabla \left(\tilde{u}_\varepsilon \circ \Phi_\varepsilon^{-1}\right)^2 \circ \Phi_\varepsilon}{2} \, dv_{\tilde{g}_\varepsilon}$$
$$+ \tilde{h}_\varepsilon(0) \int_{B(0,1)} r_\varepsilon^2 \frac{\nabla \left(\tilde{u}_\varepsilon \circ \Phi_\varepsilon^{-1}\right)^2 \circ \Phi_\varepsilon}{2} \, dv_{\tilde{g}_\varepsilon} \ .$$

Since (h_ε) converge in $C^{0,\eta}$ and Φ_ε converge in $C^{1,\eta}$, then the first term of the right-hand side goes to 0 as $\varepsilon \to 0$. Then, integrating by parts the second term, we get

$$\tilde{h}_\varepsilon(0) \int_{B(0,1)} r_\varepsilon^2 \frac{\nabla \left(\tilde{u}_\varepsilon \circ \Phi_\varepsilon^{-1}\right)^2 \circ \Phi_\varepsilon}{2} dv_{\tilde{g}_\varepsilon} = \tilde{h}_\varepsilon(0) \int_{\partial B(0,1)} r_\varepsilon^2 \frac{\tilde{u}_\varepsilon^2}{2} \overline{\nu}_\varepsilon \circ \Phi_\varepsilon d\sigma_{\tilde{g}_\varepsilon}$$

which clearly goes to 0 as $\varepsilon \to 0$. Finally, collecting the above informations, and passing to the limit $\varepsilon \to 0$ in (2.26), we get that $\nabla b(0) = 0$ if the convergence of (h_ε) holds in $C^{0,\eta}$, which achieves the proof of the Claim. ■

We are now in position to end the proof of proposition 2.1. If $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$ then we deduce the proposition from claims 2.3 and 2.4. If $\rho_\varepsilon \not\to 0$ as $\varepsilon \to 0$, then claims 2.3 and 2.4 give that $r_\varepsilon \not\to 0$ as $\varepsilon \to 0$. Then, using the Harnack inequality (2.15), one can extend the result of Claim 2.2 to $B(x_\varepsilon, 2\rho_\varepsilon) \setminus \{x_\varepsilon\}$, which proves the first part of Proposition 2.1 when $\rho_\varepsilon \not\to 0$, and ends the proof of the whole proposition.

3. Proof of theorem 3 and stability of the Pohožaev obstruction

In this section, we prove successively theorem 3 and theorem 1.

Proof of theorem 3:

We can assume that

$$\|u_\varepsilon\|_\infty \to +\infty \text{ as } \varepsilon \to 0 \ . \tag{3.1}$$

Indeed, if (u_ε) is uniformly bounded, thanks to elliptic theory, u_ε converge in $C^1(\overline{\Omega})$ to a solution u_0 of the limit equation, which prove the theorem.

Then the sequence (u_ε) develops some concentration phenomena. First in claim in Claim 3.1, as in [3], we exhaust a family of critical points of u_ε, $(x_{1,\varepsilon}, \dots, x_{N_\varepsilon,\varepsilon})$, such that each sequence $(x_{i_\varepsilon,\varepsilon})$ satisfies the assumptions of Section 2 with

$$\rho_\varepsilon = \min_{1 \le i \le N_\varepsilon, i \ne i_\varepsilon} \{|x_{i,\varepsilon} - x_{i_\varepsilon,\varepsilon}|, d(x_{i_\varepsilon,\varepsilon}, \partial\Omega)\} \ .$$

In Claim 3.2, we prove that these concentration points are in fact isolated. In other words, we prove that (u_ε) develops only finitely many concentration points, which will achieved the proof of theorem 3.

Claim 3.1. *There exists $D > 0$ such that for all $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}^*$ and N_ε critical points of u_ε, denoted by $(x_{1,\varepsilon}, \dots, x_{N_\varepsilon})$ such that :*

$$d(x_{i,\varepsilon}, \partial\Omega) u_\varepsilon(x_{i,\varepsilon})^2 \ge 1 \text{ for all } i \in [1, N_\varepsilon] \ ,$$
$$|x_{i,\varepsilon} - x_{j,\varepsilon}| u_\varepsilon(x_{i,\varepsilon})^2 \ge 1 \text{ for all } i \ne j \in [1, N_\varepsilon] \ ,$$

and

$$\left(\min_{i\in[1,N_\varepsilon]}|x_{i,\varepsilon}-x|\right)u_\varepsilon(x)^2\leq D$$

for all $x\in\Omega$ and all $\varepsilon>0$.

We define

$$d_\varepsilon=\min\left\{d(x_{i,\varepsilon},x_{j,\varepsilon}),d(x_{i,\varepsilon},\partial\Omega)\text{ s.t. }1\leq i<j\leq N_\varepsilon\right\}$$

and prove :

Claim 3.2. *There exists $d>0$ such that $d_\varepsilon\geq d$.*

Proof of Claims 3.1 and 3.2: see [3], the fact that metric and the boundary are perturbed make no difference. ■

Thanks to Claims 3.1 and 3.2, there exist $D>0,N\in\mathbb{N}^*$ and N local maxima of u_ε, $x_{1,\varepsilon},\dots,x_N$, such that:

$$d(x_{i,\varepsilon},\partial\Omega)u_\varepsilon\left(x_{i,\varepsilon}\right)^2\geq 1\text{ for all }i\in[1,N]\ ,$$
$$|x_{i,\varepsilon}-x_{j,\varepsilon}|u_\varepsilon(x_{i,\varepsilon})^2\geq 1\text{ for all }i\neq j\in[1,N]$$

and

$$\left(\min_{i\in[1,N]}|x_{i,\varepsilon}-x|\right)u_\varepsilon(x)^2\leq D$$

for all $x\in\Omega$. We can assume that $u_\varepsilon(x_{i,\varepsilon})\to+\infty$ as $\varepsilon\to 0$. Indeed, otherwise we can remove $x_{i,\varepsilon}$ from the family of concentration points, and up to changing D, the assertion remains true. Moreover, the fact that the critical point of Claim 3.1 are in fact local maxima comes from Claim 2.1. Then, thanks to Harnack inequality, there exists $C>0$ such that

$$\frac{1}{C}u_\varepsilon(x_{1,\varepsilon})\leq u_\varepsilon(x_{i,\varepsilon})\leq Cu_\varepsilon(x_{1,\varepsilon})\ . \tag{3.2}$$

Now, thanks to the results of section 2, standard elliptic theory and the fact that $\Delta+h$ is coercive, we have that, after passing to a subsequence,

$$u_\varepsilon(x_{1,\varepsilon})u_\varepsilon(x)\to u_0+\sum_{i=1}^N\lambda_i\mathcal{G}_h(x_i,x)\text{ in }C^1_{loc}(\Omega\setminus\{x_1,\dots,x_N\})\text{as }\varepsilon\to 0$$

where u_0 is a solution of the limit problem and $\mathcal{G}_h$ is the Green function of the limit operator $\Delta+h$ with Dirichlet boundary condition on Ω. ■

Proof of theorem 1:
Thanks to the remark of section 1 we can apply theorem 1 to $u_\varepsilon\circ\Phi_\varepsilon$. We can also assume that

$$\|u_\varepsilon\|_\infty\to+\infty\text{ as }\varepsilon\to 0\ . \tag{3.3}$$

else $\frac{u_\varepsilon}{\|u_\varepsilon\|_\infty}$ will converge to a positive solution of the limit problem which is impossible since the limit data satisfiePohožaev obstruction. Then with the notation of the proof of theorem 3, we have

$$u_\varepsilon(x_{1,\varepsilon})u_\varepsilon(x)\to\sum_{i=1}^N\lambda_i\mathcal{G}_h(x_i,x)\text{ in }C^1_{loc}(\Omega\setminus\{x_1,\dots,x_N\})\text{as }\varepsilon\to 0$$

Thanks to (3.2), we know that $\lambda_i > 0$ for $1 \le i \le N$. This can be rewritten as

$$G(x) = \frac{\lambda_i}{\omega_2 |x - x_i|} + G_i(x) \tag{3.4}$$

where G_i is a continuous function on $\Omega \setminus \{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N\}$. Thanks to lemma A.2 of the appendix, we can write

$$G_i(x) = G_i(x_i) + \frac{h(x_i)}{2\omega_2}|x - x_i| + \gamma_i(x) \tag{3.5}$$

where $\gamma_i \in C^1(\Omega)$ and $\gamma_i(0) = 0$. We claim that

$$G_i(x_i) = 0 \text{ for all } 1 \le i \le N\ . \tag{3.6}$$

In order to prove this, we apply the Pohožaev identity (A.1) to u_ε on the ball $B(x_{i,\varepsilon}, \delta)$ for some $\delta > 0$ small enough. Which gives, as in (2.25),

$$\begin{aligned}
&\frac{1}{2}\int_{B(x_{i,\varepsilon},\delta)} \left(h_\varepsilon u_\varepsilon^2 + h_\varepsilon \langle \Phi_\varepsilon(x) - \Phi_\varepsilon(x_{i,\varepsilon}), \nabla(u_\varepsilon \circ \Phi_\varepsilon^{-1})^2 \circ \Phi_\varepsilon\rangle\right) dv_{g_\varepsilon} = \\
&\int_{\partial B(x_{i,\varepsilon},\delta)} \langle \Phi_\varepsilon(x) - \Phi_\varepsilon(x_{i,\varepsilon}), \nabla(u_\varepsilon \circ \Phi_\varepsilon^{-1})^2 \circ \Phi_\varepsilon\rangle \left(\partial_{\nu_\varepsilon} u_\varepsilon\right)^2 + \frac{1}{2} u_\varepsilon \partial_{\nu_\varepsilon} u_\varepsilon \, d\sigma_\varepsilon \\
&- \int_{\partial B(x_{i,\varepsilon},\delta)} \frac{\langle \Phi_\varepsilon(x) - \Phi_\varepsilon(x_{i,\varepsilon}), \overline{\nu}_\varepsilon \circ \Phi_\varepsilon\rangle}{2} |\nabla(u_\varepsilon \circ \Phi_\varepsilon^{-1}) \circ \Phi_\varepsilon|^2 \, d\sigma_\varepsilon \\
&+ \left(\frac{\mu_\varepsilon}{r_\varepsilon}\right)^2 \int_{\partial B(x_{i,\varepsilon},\delta)} \langle \Phi_\varepsilon(x) - \Phi_\varepsilon(x_{i,\varepsilon}), \overline{\nu}_\varepsilon \circ \Phi_\varepsilon\rangle \frac{u_\varepsilon^6}{6} \, d\sigma_\varepsilon,
\end{aligned}$$

where $\overline{\nu}_\varepsilon$ is the exterior normal of $\Phi_\varepsilon(B(x_{i,\varepsilon}, \delta))$ and ν_ε and $d\sigma_\varepsilon$ are respectively the exterior normal and the volume element induced by g_ε on $\partial B(x_{i,\varepsilon}, \delta)$.

Thanks to the fact that h_ε is bounded, the convergence of Φ_ε ti Id and Proposition 2.1., we get the uniform estimate

$$u_\varepsilon(x_{i,\varepsilon})^2 \left| \frac{1}{2}\int_{B(x_{i,\varepsilon},\delta)} \left(h_\varepsilon u_\varepsilon^2 + h_\varepsilon \langle \Phi_\varepsilon(x) - \Phi_\varepsilon(x_{i,\varepsilon}), \nabla(u_\varepsilon \circ \Phi_\varepsilon^{-1})^2 \circ \Phi_\varepsilon\rangle\right) dv_{g_\varepsilon} \right| \le O(\delta).$$

Using (3.4), we get that

$$\begin{aligned}
&\int_{\partial B(x_{i,\varepsilon},\delta)} \left(\delta \left(\partial_\nu u_\varepsilon\right)^2 - \delta \frac{|\nabla u_\varepsilon|^2}{2} + \frac{1}{2} u_\varepsilon \partial_\nu u_\varepsilon\right) d\sigma + \int_{\partial B(x_{i,\varepsilon},\delta)} \frac{\delta}{6} u_\varepsilon^6 \, d\sigma \\
&\quad = u_\varepsilon(x_{i,\varepsilon})^{-2} \int_{\partial B(x_i,\delta)} \left(\delta \left(\partial_\nu G\right)^2 - \delta \frac{|\nabla G|^2}{2} + \frac{1}{2} G \partial_\nu G\right) d\sigma + o\left(u_\varepsilon(x_{i,\varepsilon})^{-2}\right)\ .
\end{aligned}$$

Using (3.5), we easily get that

$$\int_{\partial B(x_i,\delta)} \left(\delta \left(\partial_\nu G\right)^2 - \delta \frac{|\nabla G|^2}{2} + \frac{1}{2} G \partial_\nu G\right) d\sigma = -\frac{1}{2}\lambda_i G_i(x_i) + o(1) \text{ as } \delta \to 0\ .$$

Collecting the above informations, we prove (3.6).

We are going to prove now that $\nabla \gamma_i(x_i) = 0$ where γ_i is as in (3.5). This will contradict lemma A.3 of appendix A.3 and will achieve the proof of the theorem.

For that purpose, we apply the Pohožaev identity (A.4) to u_ε on the ball $B\left(x_{i,\varepsilon},\delta\right)$ for some $\delta>0$ small enough.

$$\begin{aligned}&\int_{\partial B(x_{i,\varepsilon},\delta)}\left(\frac{|\nabla(u_\varepsilon\circ\Phi_\varepsilon^{-1})\circ\Phi_\varepsilon|}{2}(\overline{\nu_\varepsilon}\circ\Phi_\varepsilon)-\partial_{\nu_\varepsilon}u_\varepsilon\nabla(u_\varepsilon\circ\Phi_\varepsilon^{-1})\circ\Phi_\varepsilon\right)d\sigma_\varepsilon\\&=-\int_{B(x_{i,\varepsilon},\delta)}h_\varepsilon\frac{\nabla(u_\varepsilon\circ\Phi_\varepsilon^{-1})^2\circ\Phi_\varepsilon}{2}\,dv_{g_\varepsilon}-\int_{\partial B(x_{i,\varepsilon},\delta)}\frac{u_\varepsilon^6}{6}(\overline{\nu_\varepsilon}\circ\Phi_\varepsilon)\,d\sigma_\varepsilon\ ,\end{aligned}\tag{3.7}$$

where $\overline{\nu}_\varepsilon$ is the exterior normal of $\Phi_\varepsilon(B(x_{i,\varepsilon},\delta))$ and ν_ε and $d\sigma_\varepsilon$ are respectively the exterior normal and the volume form of $\partial B(x_{i,\varepsilon},\delta)$ with respect to g_ε. It is clear that we can pass to the limit in the left-hand side. Moreover, thanks to (3.6) and (3.5), we have that

$$\int_{\partial B(x_i,\delta)}\left(\frac{|\nabla G|}{2}\nu-\nabla G\partial_\nu G\right)d\sigma\to\nabla\gamma_i(x_i)\text{ as }\delta\to0.$$

Now we look at the right-hand side of (3.7). It is clear that

$$u_\varepsilon\left(x_{i,\varepsilon}\right)^2\int_{\partial B(x_{i,\varepsilon},\delta)}u_\varepsilon^6(\overline{\nu_\varepsilon}\circ\Phi_\varepsilon)\,d\sigma_\varepsilon\to0\text{ as }\varepsilon\to0\ .$$

Then we write that

$$\begin{aligned}&\int_{B(x_{i,\varepsilon},\delta)}h_\varepsilon\frac{\nabla(u_\varepsilon\circ\Phi_\varepsilon^{-1})^2\circ\Phi_\varepsilon}{2}\,dv_{g_\varepsilon}\,dv_{g_\varepsilon}=\\&\int_{B(x_{i,\varepsilon},\delta)}(h_\varepsilon-h_\varepsilon(x_{i,\varepsilon}))\frac{\nabla(u_\varepsilon\circ\Phi_\varepsilon^{-1})^2\circ\Phi_\varepsilon}{2}\,dv_{g_\varepsilon}\\&+h_\varepsilon(x_{i,\varepsilon})\int_{B(x_{i,\varepsilon},\delta)}\frac{\nabla(u_\varepsilon\circ\Phi_\varepsilon^{-1})^2\circ\Phi_\varepsilon}{2}\,dv_{g_\varepsilon}\ .\end{aligned}\tag{3.8}$$

Since the convergence of h_ε to h holds in $C^{0,\eta}$, it is clear that the first term of the right-hand side goes to 0 as $\varepsilon\to0$. Integrating by parts the second term, we get that

$$\begin{aligned}h_\varepsilon(x_{i,\varepsilon})\int_{B(x_{i,\varepsilon},\delta)}\frac{\nabla(u_\varepsilon\circ\Phi_\varepsilon^{-1})^2\circ\Phi_\varepsilon}{2}dv_{g_\varepsilon}&=h_\varepsilon(x_{i,\varepsilon})\int_{\partial B(x_{i,\varepsilon},\delta)}\frac{u_\varepsilon^2}{2}(\overline{\nu_\varepsilon}\circ\Phi_\varepsilon)\,d\sigma_{g_\varepsilon}\\&\to h\left(x_i\right)\int_{\partial B(x_i,\delta)}\frac{G^2}{2}\nu d\sigma\end{aligned}\tag{3.9}$$

as $\varepsilon\to0$. It is easily checked that the above goes to 0 as $\delta\to0$.

Finally, collecting the above informations, and passing consecutively to the limit $\varepsilon\to0$ and $\delta\to0$ in (3.7), we get that $\nabla\gamma_i(x_i)=0$ for all i, which achieves the proof of theorem 1 thanks to lemma A.3. ■

4. Counterexample

In this section, we construct a non-stars shaped domain which admits a positive solution, moreover this domain will be as closed as we want to a ball in any $C^{0,\alpha}$-topology to the unit ball. The main idea is to glue a little domain on which there exists a solution. This will done controlling $C^{0,1}$-norm with respect the unit ball, which is an almost optimal bounded since such a construction is impossible controlling the $C^{1,\alpha}$-topology, for any α, thanks to theorem 1. We will follow the

idea of [6] and we refers to it for details.

Proof of theorem 2:

We start by defining the domain that we will glue to $B(0,1)$, in fact we will glue a fixed domain to $B\left(0,\frac{1}{\varepsilon}\right)$ and then rescale the ambient space. Hence the none star shaped domain which will glue is defined as follow

$$P_{\varepsilon,r_1}=\left\{(x,y,z)\text{ s.t. }\frac{1}{\varepsilon}-\varepsilon\leq x\leq\frac{1}{\varepsilon}+1\text{ and }y^2+z^2\leq\frac{1}{4}\right\}\setminus(B_\varepsilon\cup C_{\varepsilon,r_1}),$$

where

$$B_\varepsilon=B\left(\left(\frac{1}{\varepsilon}+\frac{1}{2},0,0\right),\frac{1}{4}\right),$$

and

$$C_{\varepsilon,r_1}=\left\{(x,y,z)\text{ s.t. }\frac{1}{\varepsilon}+\frac{1}{2}\leq x\leq\frac{1}{\varepsilon}+1;\,y^2+z^2\leq\frac{r_1^2}{4}\right\}.$$

Then we smooth the vertex of P_{ε,r_1} in order that

$$\tilde{\Omega}_{\varepsilon,r_1}=B\left(0,\frac{1}{\varepsilon}\right)\cup P_{\varepsilon,r_1},$$

be a smooth domain. Hence, we have the following picture.

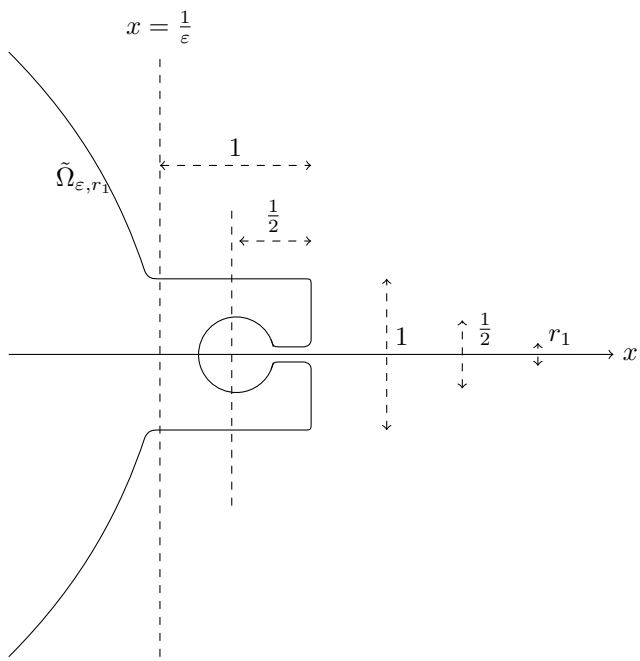

FIGURE 1. $\tilde{\Omega}_{\varepsilon,r_1}$

Then, we easily prove that there exists Φ_ε a diffeomorphism form $B\left(0,\frac{1}{\varepsilon}\right)$ to $\tilde{\Omega}_{\varepsilon,r_1}$ whose gradient's norm can be bounded independently of ε.

Finally we set,

$$\Omega_{\varepsilon,r_1} = \varepsilon\tilde{\Omega}_{\varepsilon,r_1}.$$

Then, Ω_{ε,r_1} is diffeomorphic to $B(0,1)$ via $\varepsilon\Phi_\varepsilon\left(\frac{\cdot}{\varepsilon}\right)$, which is bounded, independently of ε, in $W^{1,1}(B(0,1))$. Hence the end of this section is devoted to prove that $\Delta u = u^5$ admits a positive solution with Dirichlet boundary data on Ω_{ε,r_1} when r_1 is taken small enough.

To reach that goal, we are going to minimize the functional $E(u) = \int_{\Omega_{\varepsilon,r_1}} |\nabla u|^2\, dv$ over

$$H^* = \left\{ u \in H^1_{s,0}(\Omega_{\varepsilon,r_1}) \text{ s.t. } \int_{\Omega_{\varepsilon,r_1}} u^6\, dv = 1 \text{ and } g(u) \geq 1 + \frac{\varepsilon}{2} \right\}$$

where

$$H^1_{s,0}(\Omega_{\varepsilon,r_1}) = \left\{ u \in H^1_0(\Omega_{\varepsilon,r_1}) \text{ s.t. } u(x,,T(y,z)) = u(x,y,z) \text{ a.e. for all } T \in O(2) \right\}$$

and

$$g(u) = \int_{\Omega_\varepsilon} x|u|^6\, dv.$$

This is the space of H^1_0 function which are radial with respect to the x-axis and whose first coordinate of the " center of mass" is greater than $1 + \frac{\varepsilon}{2}$. In particular their center of mass doesn't lie in Ω_{ε,r_1}. It will be also convenient to consider the subspace of functions of H^* whose first coordinate of the " center of mass" is exactly $1 + \frac{\varepsilon}{2}$, hence we set

$$H^{1+\frac{\varepsilon}{2}} = \left\{ u \in H^1_{s,0}(\Omega_{\varepsilon,r_1}) \text{ s.t. } \int_{\Omega_{\varepsilon,r_1}} u^6\, dv = 1 \text{ and } g(u) = 1 + \frac{\varepsilon}{2} \right\}$$

Step 1:

$$S^* = \inf\{E(u) \textbf{ s.t. } u \in H^*\} > S$$

and

$$S^{1+\frac{\varepsilon}{2}} = \inf\{E(u) \textbf{ s.t. } u \in H^{1+\frac{\varepsilon}{2}}\} > S$$

where and S is the inverse square of the best constant of the Sobolev embedding of H^1_0 into L^6.

In fact, it is well known that S is independent of the domain and it is achieved only on $\mathbb{R}^3$, see chapter 1 of [8]. Then S is achieved by the following family of functions

$$U_{y,\lambda}(x) = C\frac{\lambda^{\frac{1}{2}}}{\left(\lambda^2 + \frac{|x-y|^2}{3}\right)^{\frac{1}{2}}}$$

where $y \in \mathbb{R}^3$ and λ is a positive real. For the rest of this section, we fix the constant C in dormer to $\|U_{\lambda,y}\|_6 = 1$.

The proof of Step 1 relies on the concentration compactness principle of Lions, see [5] or chapter 1 of [8], which assert that if we have $U_n \in H^1_0(\Omega)$ such that $\|U_n\|_6 = 1$ and

$$\int_{\Omega_{\varepsilon,r_1}} |\nabla U_n|^2\, dv \to S \text{ as } \to +\infty,$$

then it should exists to sequences $x_n \in \Omega$ and $\lambda_n \in \mathbb{R}_+$ such that

$$\|u_n - U_{x_n,\lambda_n}\|_6 \to 0,$$

but here by symmetry the x_n should be on the x-axis with $g(x_n) > 1 + \frac{\varepsilon}{2}$ that is to say not in Ω_{ε,r_1}, which is a contradiction and proved the first step.

Step 2: For r_1 small enough, there exists $V_\varepsilon \in H^*$ such that $E(V_\varepsilon) < min\left(2S, S^{1+\frac{\varepsilon}{2}}\right)$.

Indeed, let $y_\varepsilon = \left(1 + \frac{7\varepsilon}{8}, 0, 0\right)$, then considering $U_\lambda = U_{y_\varepsilon,\lambda}$, by definition we have, for λ small enough,

$$E\left(\frac{U_{\lambda|B\left(y_\varepsilon,\frac{\varepsilon}{16}\right)}}{\|U_{\lambda|B\left(y_\varepsilon,\frac{\varepsilon}{16}\right)}\|_6}\right) \leq \frac{2}{3} min\left(2S, S^{1+\frac{\varepsilon}{2}}\right).$$

Let $\eta : \Omega_{\varepsilon,r_1} \to [0,1]$ a smooth radial function with respect to the x-axis such that, in cylindrical coordinates,

$$\eta(x,r,\theta) = 1 \text{ if } 1 + \frac{13\varepsilon}{16} \leq x \leq 1 + \frac{15\varepsilon}{16} \text{ and } 2r_1\varepsilon \leq r \leq \frac{\varepsilon}{4}$$

and

$$\eta(x,r,\theta) = 0 \text{ if } x \leq 1 + \frac{3\varepsilon}{4} \text{ or } x \geq 1 + \varepsilon \text{ or } r \leq r_1\varepsilon \text{ or } r \geq \frac{\varepsilon}{2}.$$

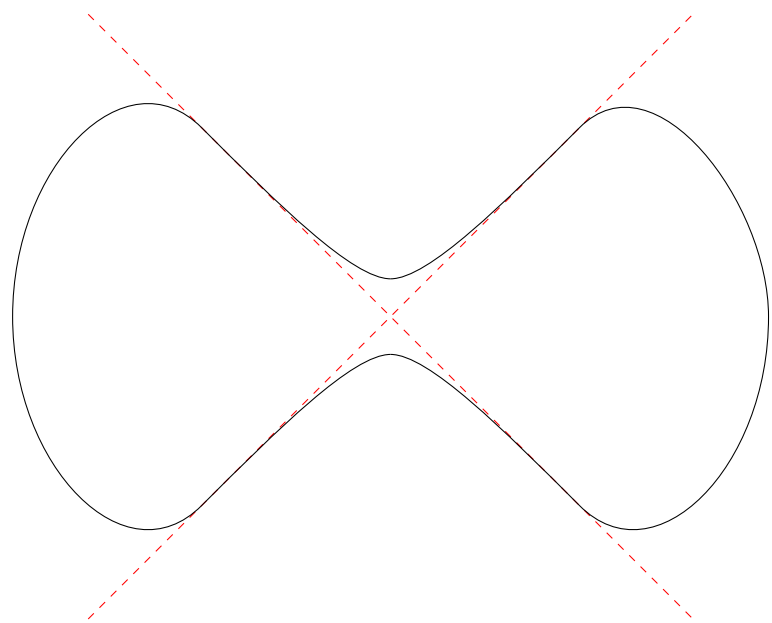

FIGURE 2. place of concentration of ηU_λ

Then $\eta U_\lambda \in H^*$ and choosing r_1 small enough, we have

$$E\left(\frac{\eta U_\lambda}{\|\eta U_\lambda\|_6}\right) < min\left(2S, S^{1+\frac{\varepsilon}{2}}\right).$$

Hence ηU_λ is the desired function.

Step 3: S^* is achieved by $U_0 \in H^*$.

As a consequence of Step 1 and Step 2, we have $S^* \in]S, 2S[$. Then thanks to standard compactness result, see theorem 3.1 of chapter 3 of [8], we know that for any Palais-Smale sequence U_n of E on $\{u \in H_0^1(\Omega) \text{ s.t. } \|u\|_6 = 1\}$ there exists $u_0 \in H_1(\Omega)$ a critical point of E under the constraint to be in H^* and some sequences $a_1^n, \cdots, a_k^n \in \Omega$ and $\lambda_1^n, \cdots, \lambda_k^n \in \Omega$ such that, up to a subsequence, we have

$$\left\| \nabla \left(U_n - U_0 \sum_{l=1}^{k} U_{a_l^l, \lambda_l^n} \right) \right\|_2 \to 0.$$

Then, since our minimizing sequence isa Palais-Smale sequence and since its limit energy is strictly between S and $2S$, it can't develop any bubble, and then converge strongly to $U_0 \in H^*$.

Step 4: U_0 is a solution of our problem.

Indeed, the constraint on $g(u)$ doesn't play any role in the Lagrangian multiplier since $S^* < S^{1+\frac{\varepsilon}{2}}$ and hence $1 + \frac{\varepsilon}{2} < g(U_0) < 1 + \varepsilon$. Moreover,

$$\int \langle \nabla \phi, \nabla U \rangle \, dv = \int \phi U_0^5 \, dv$$

for all $\phi \in H^*$, then we have

$$\Delta U_0 = U_0^5$$

using the fact that $\Delta^{-1}(u) \in H^*$ for any $u \in H^*$. Finally, we check that U_0 is positive up to replace U_n by $|U_n|$ and using the maximum principle.

Appendix A. Some technical results

A.1. A general simple lemma on functions.

Lemma A.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$. Let $u \in C^1(\Omega)$ be a function positive in the interior and null on the boundary. Assume that*

$$\{x \in \Omega \text{ s.t. } \nabla u(x) = 0 \text{ and } d(x, \partial\Omega) u^2(x) \geq 1\} \neq \emptyset \, .$$

Then there exist $N \in \mathbb{N}^*$ and N critical points of u, denoted by $(x_1, \ldots, x_N)$, such that

$$d(x_i, \partial\Omega) u(x_i)^2 \geq 1 \text{ for all } i \in [1, N],$$
$$|x_i - x_j| u(x_i)^2 \geq 1 \text{ for all } i \neq j \in [1, N]$$

and

$$\left(\min_{i \in [1,N]} |x_i - x| \right) u(x)^2 \leq 1$$

for all critical points x of u such that $d(x, \partial\Omega) u(x)^2 \geq 1$.

Proof of Lemma A.1: see [3]. ■

A.2. **General Pohožaev's identities.** In this section we remind some classical Pohožaev identities, see [7], we used in this paper. Assume that u is a C^2- solution of

$$\Delta u = u^5 - hu \text{ in } \Omega \ .$$

Multiplying this equation by $< x, \nabla u >$ and integrating by parts, one easily gets that

$$\frac{1}{2}\int_\Omega \left(hu^2 + h < x, \nabla u^2 >\right) dx = B, \tag{A.1}$$

where

$$B = \int_{\partial\Omega} \left(< x, \nabla u > \partial_\nu u + \frac{1}{2} u \partial_\nu u - < x, \nu > \frac{|\nabla u|^2}{2} \right) d\sigma$$
$$+ \int_{\partial\Omega} < x, \nu > \frac{u^6}{6} d\sigma \ .$$

Hence, if $u = 0$ on $\partial\Omega$, we get that

$$\frac{1}{2}\int_\Omega h\left(u^2 + < x, \nabla u^2 >\right) dx = \int_{\partial\Omega} < x, \nu > (\partial_\nu u)^2 \, d\sigma \ . \tag{A.2}$$

Integrating by parts again, we get the Pohožaev identity in its usual form :

$$\int_\Omega \left(h + \frac{< x, \nabla h >}{2} \right) u^2 dx = -\int_{\partial\Omega} < x, \nu > (\partial_\nu u)^2 \, d\sigma \ . \tag{A.3}$$

In a similar way, multiplying the equation by ∇u and integrating by parts, one can derive the following Pohožaev's identity :

$$\int_{\partial\Omega} \left(\frac{|\nabla u|^2}{2} \nu - \partial_\nu u \nabla u + \frac{u^6}{6} \nu \right) d\sigma = \int_\Omega h \frac{\nabla u^2}{2} dx \ . \tag{A.4}$$

A.3. **Pohožaev's identity for Green functions.** In this section, we prove a useful Pohožaev identity for a sum of Green's functions. First of all, we easily derive the following Lemma from standard elliptic theory :

Lemma A.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$. Let $y \in \Omega$ and let g be a weak solution in $H^1(\Omega)$ of*

$$\Delta g + hg = -\frac{h}{\omega_2 |x - y|} \ \text{ in } \Omega \ .$$

Then g is continuous and can be written as

$$g(x) = g(y) + \frac{h(y)}{2}|x - y| + \gamma_y(x) \ \text{ in } \Omega \tag{A.5}$$

where $\gamma_y \in C^1(\Omega)$ satisfies $\gamma_y(y) = 0$.

Applying the previous decomposition lemma to Green's functions, we get the following Pohožaev identity on the regular parts of them.

Lemma A.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$, star-shaped with respect to 0 and let $h \in C^1(\Omega)$ which satisfies (**??**). Let $\mathcal{G}_h$ be the Green function of $\Delta + h$. Let also $N \in \mathbb{N}^*$, $x_1, \dots, x_N \in \Omega$, $\lambda_1, \dots, \lambda_N$ some positive real numbers and*

$$G(x) = \sum_{i=1}^N \lambda_i \mathcal{G}_h(x, x_i) \ .$$

Then, using lemma A.2, we write G in a neighbourhood of x_i as

$$G(x) = \frac{\lambda_i}{\omega_2|x-x_i|} + m_i + \lambda_i \frac{h(x_i)}{2}|x-x_i| + \gamma_i(x)$$

where $m_i \in \mathbb{R}$ and $\gamma_i \in C^1(\Omega)$ satisfies $\gamma_i(0) = 0$. Then we have that

$$\sum_{i=1}^{N} \lambda_i \left(m_i + 2 < x_i, \nabla\gamma_{x_i}(x_i) >\right) < 0 \ .$$

Proof of lemma A.3: see [3]. ■

References

1. L.A. Caffarelli, B. Gidas, and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), no. 3, 271–297.
2. O. Druet and E. Hebey, *Stability and instability for Einstein-scalar field Lichnerowicz equations in compact Riemannian manifolds*, Math. Zeitschrift (to appear).
3. Olivier Druet and Paul Laurain, *Stability of the Pohožaev obstruction in dimension 3.*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 5, 1117–1149.
4. Q. Han and F. Lin, *Elliptic partial differential equations*, Courant Lecture Notes in Mathematics, vol. 1, New York University Courant Institute of Mathematical Sciences, New York, 1997.
5. Pierre-Louis Lions, *Applications de la méthode de concentration-compacité à l'existence de fonctions extrémales*, C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), no. 15, 645–648. MR 705681 (85b:49023)
6. Donato Passaseo, *Multiplicity of positive solutions of nonlinear elliptic equations with critical Sobolev exponent in some contractible domains*, Manuscripta Math. **65** (1989), no. 2, 147–165. MR 1011429 (90g:35051)
7. S. I. Pohožaev, *On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$*, Dokl. Akad. Nauk SSSR **165** (1965), 36–39.
8. Michael Struwe, *Variational methods*, fourth ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 34, Springer-Verlag, Berlin, 2008, Applications to nonlinear partial differential equations and Hamiltonian systems. MR 2431434 (2009g:49002)

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